On the Spectrum of some Gravitational Instantons

by

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Abstract

In this thesis we study Dirac operators on the Euclidean Taub-NUT and Schwarzschild spaces coupled to abelian gauge fields, with the aim of computing the zero-modes and bound states. The work is motivated by recently proposed Geometric Models of Matter, where single particles are modelled by 4-manifolds and their quantum numbers realised as topological invariants of the model manifolds. In these models, the spin degrees of freedom are given by the zero-modes of the Dirac operator.

In the case of the Taub-NUT manifold coupled to an $U(1)$ gauged field with self-dual curvature, which is the model for the electron, we are able to obtain explicit expressions for the zero modes of the Dirac operator. We show that they decompose into an irreducible representation of $SU(2)$ and use this to interpret a known index theorem in this geometry first deduced by Pope.

We also study the dynamical symmetry of this space in the classical and quantum cases, and show that the gauge field allows the existence of classical bounded orbits and quantum bound states. We compute scattering cross sections and find a surprising electric-magnetic duality. Using twistor formalism we are able to show that the dynamical symmetry is preserved in the gauged case and that this makes possible to deduce the energy of the quantum bound states in an algebraic manner.

We consider the Euclidean Schwarzschild manifold coupled to an $U(1)$ gauge field as a neutron candidate. In this case the zero-modes of the Dirac operator also decompose into an irreducible representation of $SU(2)$. Using the open code SLEIGN2, we compute the spectrum of the Laplace-Beltrami operator acting on scalar fields.

Dedication

I would like to dedicate this thesis to Ania.

Acknowledgements

I would like to thank all the people who in one way or another have contributed in making this work possible. I am specially indebted to my supervisor, Bernd Schroers, for his guidance, kindness and patience. I am grateful to have worked with such an intellectual and brilliant supervisor. I appreciate his contribution to the award of my PhD scholarship.

Contents

Chapter 1

Introduction and Motivation

1.1 Geometric Models of Matter

In recent years, geometric models for fundamental particles, henceforth referred to as GMM, were proposed [1]. In these models, static charged particles are represented by non-compact 4-manifolds and the quantum numbers of the particles are identified with topological invariants of the model manifolds. This topological description of physical quantities was motivated by a model for baryons known as the Skyrme model [2] in which there is a conserved topological charge identified with the baryon number.

The main difference between the Skyrme model and GMM of [1] is that the former is a field theory given by a Lagrangian density whose static field configurations of minimal energy gives the field of a particle, whilst in the latter a particle is represented completely by the geometry of a manifold. In the Skyrme model only the baryon number is topological and the spin and isospin follow from the quantisation of rotations of the field configuration, whereas in the geometric description all quantum numbers are proposed to be topological. Another difference is that Skyrmions only describe baryons while GMM is more ambitious in the sense that it aims for a unified description of both baryonic and leptonic particles in the same framework.

In GMM, spin-$\frac{1}{2}$ particles such as the electron, neutron and proton are modelled by 4-manifolds that, away from a core region, are a circle fibration. Whether the fibration is trivial or not, determines if the particle is neutral or charged. The Dirac operator twisted by the $U(1)$ bundle is used to model the spin degrees of freedom of the particle. The region where the fibre is ill-defined is interpreted as the boundary or the core of the particle. In a model for a point-like particle, like the electron, the fibre should then collapse in a point whilst for particles like the proton and neutron the core should have finite size.

The Taub-NUT (TN) and the Atiyah-Hitchin (AH) manifolds were initially proposed as the geometric models for the electron and proton respectively. The baryon number is identified with the signature of the manifold which is zero for TN and 1 for AH. To define the electric charge, one makes use of their compactification which is $\mathbb{CP}^2$ for both TN and AH. The electric charge is then defined as minus the self-intersection number of the "surface at infinity" which compactifies the manifold to $\mathbb{CP}^2$ ($\mathbb{CP}^1$ for TN and $\mathbb{RP}^2$ for AH). The self-intersection number can also be understood as the first Chern number of the $U(1)$ fibration. With this set up, a model for the neutron should be a manifold with signature 1 which is either non-compact and a trivial fibration or compact. The model that was initially proposed for the neutron is $\mathbb{CP}^2$ which has the right signature and resembles AH in its orbit structure [1].

In the asymptotic region, the AH manifold is a circle fibration over $\mathbb{RP}^2 \simeq S^2/\mathbb{Z}_2$. However, the identification of the opposite points of the sphere is physically problematic. An alternative to this model is the Taub-Bolt space which also has signature 1, and as the TN space, it is a circle fibration over $\mathbb{CP}^1$ which is actually the standard Hopf bundle. In this way the electric charge has the same interpretation as in TN, but the orientation of the fibres has to be reversed as to obtain an opposite sign. With the Taub-Bolt as a model for the proton, it seems natural to choose the Euclidean Schwarzschild (ES) space as the model for the neutron as both spaces have a similar geometry in the core. It also has a trivial circle fibration as required for a neutral particle, and as in Taub-Bolt, the fibre collapses on a spherical surface (known as the Bolt) which can be identified with the core of the neutron.

In this thesis we consider the Dirac operator on TN and ES spaces, twisted by the

$U(1)$ bundle of their asymptotic fibrations, as the model for the spin of the electron and neutron in GMM. We investigate whether the zero-modes of the Dirac operator can account for the spin degrees of freedom of these particles. This problem has its own intrinsic interest from the mathematical point of view. The classical and quantum dynamics of TN is an interesting problem too and has been studied before [3]. We extend this study to the gauge case and find interesting new features. The Dirac and Laplace operators on ES have not been studied to the same extent, and ours is the first detailed study of their spectrum.

1.2 Taub-NUT

The TN space has been of interest from the General Relativity point of view too, as it is an example of a gravitational instanton. For us, a gravitational instanton is a solution of the Euclidean vacuum Einstein field equations. Mathematicians have a more restrictive definition which requires self-duality of the Riemann tensor (and therefore the metric be Hyperkaehler). TN fits in both definitions as it is a Riemannian 4-manifold with self-dual curvature which implies [4] that the Euclidean vacuum Einstein equations are satisfied. This space has the topology of $\mathbb{C}^2$. Away from the origin, the TN space has the structure of a $U(1)$ bundle over $\mathbb{R}^3\backslash 0 \cong S^2$ which, as already mentioned, is identified with the Hopf Bundle. As a model for the electron the radius of the fibre at infinity is interpreted as the classical electron radius [1], [5].

In the TN model for the electron, the baryon number and electric charge are identified with two topological invariants of the manifold. The baryon number was tentatively associated with the signature which is zero for TN as it has the topology of $\mathbb{C}^2$. As we mentioned before, the electric charge is identified with minus the self-intersection number of $\mathbb{CP}^1 \cong S^2$ in $\mathbb{CP}^2$, which is the compactification of TN. This follows from the fact that $\mathbb{CP}^2 \simeq \mathbb{CP}^1 \cup \mathbb{C}^2$, which enables one to write $\mathbb{C}^2 \simeq \mathbb{CP}^2\backslash\mathbb{CP}^1$ and consequently to use $\mathbb{CP}^2$ as the compactification of TN. The self-intersection number can be interpreted [1] as the flux of a harmonic and rotationally invariant 2-form over the 2-sphere. Thus the harmonic form may be interpreted as the electric field of the electron.

Since TN is topologically trivial there is no natural normalisation of the harmonic form, but in our discussion we will fix the scale by normalising the flux. In terms of the detailed discussion of the TN space [1], we normalise the 2-form to be the Poincaré dual of $\mathbb{CP}^1$ at infinity which, as already mentioned, compactifies TN to $\mathbb{CP}^2$.

A realistic model for an electron should also account for its spin-$\frac{1}{2}$. In the TN model for the electron, the spin degrees of freedom were initially proposed [1] to be given by the zero-modes of the Dirac operator on this manifold. This is consistent with the idea of a topological description of quantum properties, as the zero-modes are topological. This follows from the Atiyah-Singer index theorem [6] which, in the non-compact case, relates the zero-eigenvalues with the Pontryagin number $\tilde{p}$, a contribution from infinity called the η-invariant and the second Chern class of the fibre bundle. For a compact manifold, if n_+ and n_- are the number of regular L^2 normalisable zero-eigenvalues with positive and negative chirality respectively, and if there is a vanishing theorem implying $n_- = 0$ then [7] $n_+ - n_- = n_+ = -\frac{1}{24}\tilde{p}$.

The aim of the project that eventually became this thesis was to explore whether the zero-modes of the Dirac operator on Euclidean 4-manifolds, which serve as a geometric models of spin-$\frac{1}{2}$ particles, contain a doublet of normalisable eigenvalues that can account for their spin degrees of freedom. In the TN as case study, the gauge potential of the harmonic 2-form, which plays the role of the electric field of the electron, has an important role too. More precisely, the Dirac operator coupled to the gauge potential has a non-trivial kernel as first pointed out by Pope [7]. The project then evolved towards the study of the classical and quantum dynamics of TN and in this study the inclusion of the gauge potential in the dynamics seemed to be natural. Indeed even while TN has neither bounded orbits nor quantum bound states the magnetic binding of the gauge field produces both.

To get intuition on the zero-modes of the Dirac operator coupled to an abelian gauge field in TN, we looked at the simpler problem of finding the zero-modes of the

Dirac operator on S^2 coupled to the Dirac monopole. This simplification is natural as it is basically the asymptotic limit (away from the origin) of the original problem. In other words, at this limit the TN geometry turns into a circle fibration over S^2 and the gauge potential of the harmonic 2-form reduces to a connection on this bundle, whose local form on S^2 turns out to be the Dirac monopole. Furthermore, this problem provides one of the simplest illustrations of an index theorem [6]. For a monopole of magnetic charge g and a spinor of electric charge e, the product of electric and magnetic charge is an integer multiple of Plank's constant by Dirac's quantisation condition, i.e.,

$$\frac{eg}{2\pi\hbar} = n \in \mathbb{Z}. \tag{1.1}$$

Mathematically, coupling to a Dirac monopole amounts to twisting the Dirac operator on S^2 by a complex line bundle with connection. The integer n is the Chern number of the line bundle and the index of the Dirac operator turns out to be n too. Together with a vanishing theorem, this gives the dimension of the space of zero-modes as $|n|$ [8, 9]. Physically, there is therefore one state per $2\pi\hbar$ cell volume in the electric-magnetic charge plane.

The index is independent of the detailed form of the magnetic field and the metric on S^2. However, by specialising to the round metric on S^2 and the magnetic monopole field, we can bring the double cover $SU(2)$ of the isometry group into the picture. The twisted Dirac operator and its kernel are now naturally acted on by $SU(2)$ and the kernel is, in fact, the irreducible $SU(2)$ representation of dimension $|n|$. Parametrising S^2 in terms of a complex coordinate via the stereographic projection, one can realise the zero-modes in terms of holomorphic (for $n \geq 0$) or antiholomorphic (for $n \leq 0$) polynomials of degree $|n| - 1$.

In this thesis we review these results and use them to gain a better understanding of an index formula due to Pope for the Dirac operator on TN, coupled to the magnetic potential of the harmonic form with a flux p,

$$\dim \ker \not{D}_p = \frac{1}{2}[|p|]([|p|] + 1), \tag{1.2}$$

where, for a positive real number x, we define $[x]$ as the largest integer strictly smaller than x [7, 10]. Here, we would like to understand the $SU(2)$ transformation properties of these zero-modes, and to gain a qualitative understanding of why the Dirac operator on TN only has zero-modes if one twists it by an abelian gauge field - even though the TN geometry already encodes a Dirac monopole.

As mentioned before, we normalise the harmonic 2-form to be the Poincaré dual of $\mathbb{CP}^1$ and so we allow its gauge potential to have the structure group $(\mathbb{R}, +)$ rather than $U(1)$ so that the unitary representation of an element $u \in \mathbb{R}$ is by a phase e^{ipu} with $p \in \mathbb{R}$. When we twist the Dirac operator with this bundle, spinors may therefore have any real charge p. On the topologically trivial TN manifold, there is not a Dirac condition like (1.1) to force the product of the electric and magnetic charge to be an integer or, equivalently, the gauge group to be $U(1)$.

The TN space already encodes a Dirac monopole whose charge s is the eigenvalue of the central $U(1)$ in the $U(2)$ isometry group. Assuming for simplicity $p > 0$, we find that zero-modes are normalisable only if s satisfies the analogue of (1.1) with $n = 2s + 1 \leq [p]$. Moreover, we learn that, for each allowed value of n, there is an n-dimensional space of zero-modes, forming an irreducible $SU(2)$ representation. The space of zero-modes is the direct sum of these representations, reproducing and interpreting Pope's dimension formula (1.2) as the sum $1 + 2 + \cdots + ([p] - 1) + [p]$. Our discussion also shows that it is possible to obtain a spin-$\frac{1}{2}$ doublet of states from the normalisable zero-modes by picking $2 < p \leq 3$ as required. However, with this choice one also obtains a spin-0 singlet, as $[p]$ only sets an upper limit on the dimensions of the irreducible $SU(2)$ representations.

The fact that the index of the Dirac operator, minimally coupled to the harmonic form, is non-trivial shows that this 2-form is intimately connected to the TN geometry. This relation can also be seen from the fact that, with a suitable normalisation, the harmonic form is the Poincaré dual of $\mathbb{CP}^1$ which compactifies TN to $\mathbb{CP}^2$. We would also like to understand the role the harmonic form plays in the eigenstates of the Laplace operator obtained by squaring the Dirac operator. In the context of

GMM the bound states of the Laplace operator can, perhaps be interpreted as the quantum modes of the particle modelled by the TN geometry. In other words, we use these operators to account for and to study the quantum dynamics of gauged TN. We are able to show that there are infinitely many bound states, giving a tower of particles with charge -1 and spin-$\frac{1}{2}$ with the electron as the ground state. The other states could then possibly be identified with higher energy particles with these quantum numbers, such as the Muon and Tau. However, this is very speculative and we do not pursue it here.

We are also interested in the classical dynamics of TN in the gauged case, which is given by the geodesic motion on this space. This problem has been studied before in the non-gauged case [3], where the classical trajectories (in the case of negative mass) were studied by means of a conserved quantity analogous to the Runge-Lenz vector of the Kepler problem. We show that in the gauged case there is also a Runge-Lenz vector which allows us to study the classical trajectories in a similar way.

All interesting algebraic features of ordinary TN dynamics carry over to the gauge case. We show that bounded motions and quantum bound states, neither of which are possible on TN alone, occur in the gauge dynamics. The reason for this is that the gauge potential produces a magnetic binding akin to that responsible for Landau levels in planar systems [11]. In fact, close to the origin where TN becomes flat Euclidean 4-space, the 4-dimensional magnetic field is constant, and the bound states that we find become ordinary Landau levels. This picture of magnetic binding also provides a qualitative explanation of the index (1.2) found by Pope.

1.3 Euclidean Schwarzschild

As mentioned before, the ES space is a candidate model for the neutron in GMM. This space has signature 1 and the topology of $\mathbb{R}^2 \times S^2$. Asymptotically, it is a trivial circle fibration over the 2-sphere which is a necessary condition for a non-charged particle in GMM. The fibre collapses on a sphere which is known as the "bolt" and can be interpreted as the core of the neutron. As in the TN case, ES admits an harmonic 2-form which can be obtained from a $U(1)$ gauge potential. This potential is a connection on a $U(1)$ bundle over ES which is topologically non-trivial, while the analogous bundle over TN $\cong \mathbb{R}^4$ is trivial as all spheres on $\mathbb{R}^4$ are contractible.

As in TN, the zero-modes of the Dirac operator on ES, twisted by a $U(1)$ gauge potential with charge p, are considered as the model for the spin degrees of freedom. In this case, the dimension of the space of zero-modes is $|p|^2$ [7], and so the degeneracy grows quadratically as in TN. The existence of zero-modes in ES for non-zero p, can again be explained in terms of the magnetic binding of the gauge potential.

By introducing complex coordinates, it is possible to identify the angular part of the zero modes with the zero-modes on the 2-sphere. This means that, as in the 2-sphere problem, the zero-modes on ES are irreducible $SU(2)$ representations of dimension $|p|$. The problem of computing the eigenfunctions of the Dirac operator is also considered, but in this case such computation is much more difficult. However, numerical solutions can be obtained for the Laplace operator acting on scalar fields. We do not discuss the classical trajectories as in the TN case because ES does not admit a conserved Runge-Lenz vector that can be used to simplify this task.

1.4 Overview

Chapter 2 of the thesis contains background material outlining the Dirac operator and Lens spaces. We review the definition of the Dirac operator coupled to a gauge potential on a Riemannian manifold with spin connection. We discuss vector fields on the Lens spaces as well as sections of line bundles associated to these spaces, and review how the latter transform under the $SU(2)$ action. We consider the Dirac monopole as the local form of a connection 1-form on the Lens spaces.

With this framework established, we review in Chapter 3 the zero-modes of the

Dirac operator coupled to the Dirac monopole, first on S^2 and then on $\mathbb{R}^3$ with a suitable mass term, induced by dimensional reduction. Moving on to the case of the twisted Dirac operator on TN we are able to compute the explicit form of its zero-modes and to show that they form irreducible representations of $SU(2)$.

In Chapter 4 we review the notion of symmetry in classical mechanics and, as an example, we consider the Kepler problem. We use the compactification of momentum space to S^3 [12] to show that the associated angular momentum and Runge-Lenz vectors are the moment maps of a $SO(4)$ action on S^3. In order to gain intuition on the quantum dynamics on gauged TN, we consider a toy model of the motion on a surface, which shows how a magnetic field on the surface can produce bound states. We discuss the classical dynamics on TN and use the conserved angular momentum and Runge-Lenz to describe the gauged classical trajectories.

Moving on to the quantum case, we use separation of variables to solve the eigenvalue problem of the Laplace operator coupled to the gauge potential on TN. We exhibit the bound states, give their energies and degeneracies and compute scattering cross sections. We show that this problem can also be solved algebraically, using a quantum version of the gauged Runge-Lenz vector. We end the chapter by exhibiting the symmetry underlying the conservation of angular momentum and Runge-Lenz vectors from a twistorial description of phase space. The main results of Chapters 3 and 4 have been published in references [13, 14].

In Chapter 5 we derive the Euclidean Schwarzschild solution of the vacuum Einstein field equations by using the tetrad method. We compute the Dirac operator associated to this metric and minimally couple it to an abelian gauge field of charge p, which is the potential of an harmonic form. We explicitly compute the normalised zero-modes in both spherical and complex coordinates and show that they span a space of dimension $|p|^2$. We are able to solve numerically the eigenvalue problem of the gauged Laplace-Beltrami operator acting on scalar fields by using the open code SLEIGN2. Chapter 6 concludes the thesis with a discussion of the main results.

Chapter 2

Background

2.1 Dirac operator

In this chapter we introduce the objects and techniques used throughout the thesis. This involves concepts of differential geometry such as vector fields and differential forms on manifolds, which are commonly used in the description of Einstein's theory of gravity. We also discuss the ideas of spin connection, Dirac operators and the Hopf bundle which is intimately related to the Dirac operators considered in this thesis.

In the 1920's P.A.M. Dirac, not satisfied with the relativistic description of the electron, was searching for a first order relativistic equation for the electron that was compatible with the Klein-Gordon equation. So he was essentially looking for a first order differential equation whose square was the Laplacian. Dirac realised that the operator

$$\not{D} = \gamma^j \partial_j, \tag{2.1}$$

where sum over repeated indices is assumed, [1] has this property provided that the γ^i's are matrices satisfying the generating relations of a Clifford algebra:

$$\gamma^i \gamma^j + \gamma^j \gamma^i = -\delta_{ij}. \tag{2.2}$$

The Dirac equation has been of great relevance in physics as it was the first theory to

[1] In the future we will always use this convention unless otherwise stated.

account for relativistic particles of spin-$\frac{1}{2}$ such as electrons and quarks, and predicted the existence of an electron-like particle with positive electric charge. The Dirac operator has been of mathematical interest too; it has helped to rediscover the Laplace operator, as it can be realised as the square of the Dirac operator, and has led to important results in modern mathematics such as the Atiyah-Singer index theorem.

2.1.1 Dirac operator in a curved space

The definition of the Dirac operator (2.1) is only valid in a flat space but it can can be generalised to a curved manifold. In this section we show this generalisation avoiding going through the formal mathematical construction. A detailed definition of Dirac operators in a more general context can be found in [15, 16].

As we know, a metric of a Riemannian manifold can be written in terms of an orthonormal basis [2] $e^i_\mu = e^i_\mu dx^\mu$ of the cotangent space, as follows:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \equiv e^i_\mu e^i_\nu dx^\mu dx^\nu. \tag{2.3}$$

The dual vector fields E_j to e^i, defined by the relation $(e^i, E_j) = \delta^i_j$, form an orthonormal basis of the tangent space: $E_j = E^\mu_j \partial_\mu$. The inverse of the metric is given by $g^{\alpha\beta} = E^\alpha_j E^\beta_j$ since

$$g_{\mu\alpha} g^{\alpha\beta} = e^i_\mu e^i_\alpha E^\alpha_j E^\beta_j = e^j_\mu E^\beta_j = \delta^\beta_\mu. \tag{2.4}$$

With this set up, we can generalise the Dirac operator by replacing the flat γ-matrices (2.2) by the curved version of them given by contraction with the components of the dual frame:

$$\gamma^\mu = \gamma^i E^\mu_i. \tag{2.5}$$

Observe that the curved γ-matrices satisfy

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = (\gamma^i \gamma^j + \gamma^j \gamma^i) E^\mu_i E^\nu_j = -2\delta^{ij} E^\mu_i E^\nu_j = -2g^{\mu\nu}. \tag{2.6}$$

[2] As it is usually the convention we use Greek letters for coordinates indices.

We also replace the ordinary partial derivatives by covariant derivatives associated to the spin connection

$$\tfrac{\partial}{\partial x^\mu} \rightarrow \tfrac{\partial}{\partial x^\mu} + \Gamma_\mu, \tag{2.7}$$

where $\Gamma_\mu = \Gamma(\partial_\mu)$, and the spin connection 1-form Γ is defined by the no torsion equation

$$de + [\Gamma, e] = 0. \tag{2.8}$$

Here e is the γ-valued differential form $e = e^i\gamma^i$ and so the second term is obtained by combining the exterior product of forms with the usual multiplication of matrices. Then expressing

$$\Gamma = -\frac{1}{8}[\gamma_i, \gamma^j]\omega^i{}_j, \tag{2.9}$$

in terms of components, equation (2.8) becomes

$$de^i\gamma^i - \frac{1}{8}\left[[\gamma_i, \gamma^j], \gamma^k\right]\omega^i{}_j \wedge e^k = 0. \tag{2.10}$$

Finally using the relation

$$\left[[\gamma^i, \gamma^j], \gamma^k\right] = 4(\delta^{ik}\gamma^j - \delta^{jk}\gamma^i), \tag{2.11}$$

we find that the no torsion equation is equivalent to

$$de^i + \omega^i{}_k \wedge e^k = 0, \quad \omega_{ij} = -\omega_{ij}. \tag{2.12}$$

The curvature 2-form is given by the covariant derivative of the spin connection:

$$R^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j, \tag{2.13}$$

Defining $R^i{}_{jkl} = R^i{}_j(E_k, E_l)$ so that $R^i{}_j = R^i{}_{jkl}e^k \wedge e^l$, it follows [17] that the components $R^i{}_{jkl}$ are related to the curvature $R^\alpha{}_{\beta\mu\nu}$ of the Levi-Civita connection

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\delta}(g_{\mu\delta,\nu} + g_{\nu\delta,\mu} - g_{\mu\nu,\delta}) \tag{2.14}$$

by the relation

$$R^\alpha{}_{\beta\mu\nu} = E^\alpha_i e^j_\beta e^k_\mu e^l_\nu R^i{}_{jkl}. \tag{2.15}$$

In terms of the curved γ-matrices, the generalised Dirac operator reads

$$\not{D} = \gamma^j E^\mu_j (\tfrac{\partial}{\partial x^\mu} + \Gamma_\mu) \equiv \gamma^\mu D_\mu. \tag{2.16}$$

In the future we will consider the curved Dirac operator coupled to a $U(1)$ connection A_μ,

$$\not{D}_A = \gamma^j E^\mu_j (\tfrac{\partial}{\partial x^\mu} + \Gamma_\mu + A_\mu). \tag{2.17}$$

2.1.2 Lichnerowicz theorem

In a flat Euclidean space with metric $g^{\mu\nu}$ the Dirac operator (2.1) squares to minus the Laplace operator

$$\not{D}^\dagger \not{D} = -g^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\mu} = -\frac{\partial^2}{\partial t^2} - \nabla^2. \tag{2.18}$$

In the case of a Lorentzian space with metric $\eta^{\mu\nu} = (-,+,+,+)$ one gets the Klein-Gordon operator instead. In a curved space the square of the Dirac operator has a similar expression with an extra term proportional to the scalar curvature. This result, known as Lichnerowicz theorem, has been discussed in several books see for example [16, 18]. For pedagogical reasons we now show a proof obtained by an explicit calculation. So using the expression (2.16) for the curved Dirac operator we see that

$$\begin{aligned} \not{D}^\dagger \not{D} &= \gamma^\mu D_\mu \gamma^\nu D_\nu \\ &= \gamma^\mu \frac{\partial \gamma^\nu}{\partial x^\mu} D_\nu + \gamma^\mu \gamma^\nu D_\mu D_\nu. \end{aligned} \tag{2.19}$$

It is possible to work in a coordinate system where the the partial derivatives of the curved γ-matrices vanish [18], and so in this case we have

$$\begin{aligned} \not{D}^\dagger \not{D} &= \frac{1}{2}\gamma^\mu \gamma^\nu D_\mu D_\nu + \frac{1}{2}\gamma^\mu \gamma^\nu D_\mu D_\nu \\ &= -g^{\mu\nu} D_\mu D_\nu + \frac{1}{2}\gamma^\mu \gamma^\nu [D_\mu, D_\nu] \\ &= -g^{\mu\nu} D_\mu D_\nu + \frac{1}{4}[\gamma^\mu, \gamma^\nu][D_\mu, D_\nu], \end{aligned} \tag{2.20}$$

where we have used the relation (2.6). We now show that the commutator of the covariant derivatives gives the curvature tensor (2.13). To see this we use (2.9) to compute

$$
\begin{aligned}
[D_\mu, D_\nu] &= \frac{1}{8}[\gamma_i, \gamma^j](\partial_\nu \omega^i{}_{j\mu} - \partial_\mu \omega^i{}_{j\nu}) \\
&\quad + \frac{1}{64}[\gamma_i, \gamma^j][\gamma_k, \gamma^l]\omega^i{}_{j\mu}\omega^k{}_{l\nu} - \frac{1}{64}[\gamma_i, \gamma^j][\gamma_k, \gamma^l]\omega^i{}_{j\nu}\omega^k{}_{l\mu}.
\end{aligned} \tag{2.21}
$$

The last two terms can be simplified by using the following relation [19] for the γ-matrices:

$$
\begin{aligned}
[\gamma^i, \gamma^j][\gamma^k, \gamma^l] &= 2\delta^{jl}[\gamma^i, \gamma^k] - 2\delta^{il}[\gamma^j, \gamma^k] + 2\delta^{ik}[\gamma^j, \gamma^l] - 2\delta^{jk}[\gamma^i, \gamma^l] \\
&\quad - 4(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}) + 4\epsilon^{ijkl}\gamma^1\gamma^2\gamma^3\gamma^4.
\end{aligned} \tag{2.22}
$$

Then we obtain

$$
\begin{aligned}
[D_\mu, D_\nu] &= -\frac{1}{8}[\gamma_k, \gamma^l](\partial_\mu \omega^k{}_{l\nu} - \partial_\nu \omega^k{}_{l\mu} + \omega^k{}_{m\mu}\omega^m{}_{l\nu} - \omega^k{}_{m\nu}\omega^m{}_{k\mu}) \\
&= -\frac{1}{8}[\gamma_k, \gamma^l]R^k{}_{l\mu\nu}.
\end{aligned} \tag{2.23}
$$

Inserting this into the expression for $\not{D}^\dagger \not{D}$ we find

$$
\begin{aligned}
\not{D}^\dagger \not{D} &= -g^{\mu\nu}D_\mu D_\nu - \frac{1}{32}E^\mu_i E^\nu_j [\gamma^i, \gamma^j][\gamma_k, \gamma^l]R^k{}_{l\mu\nu} \\
&= -g^{\mu\nu}D_\mu D_\nu - \frac{1}{32}[\gamma^i, \gamma^j][\gamma_k, \gamma^l]R^k{}_{lij} \\
&= -g^{\mu\nu}D_\mu D_\nu + \frac{1}{4}R^k{}_{lkl} + \frac{1}{4}R^l{}_{klj}[\gamma^k, \gamma^j] - \frac{1}{8}\gamma^1\gamma^2\gamma^3\gamma^4\epsilon^{ijkl}R_{ijkl},
\end{aligned} \tag{2.24}
$$

where we have used again the identity (2.22) in the last step. Note that the second term of the right hand side is the scalar curvature $R = R^k{}_{lkl}$ and the third one is the Ricci tensor $R_{kj} = R^l{}_{klj}$. Because of the symmetry of the Ricci tensor, the third term is zero. The fourth term is also zero by the Bianchi identity, and so altogether

$$
\not{D}^\dagger \not{D} = -g^{\mu\nu}D_\mu D_\nu + \frac{1}{4}R. \tag{2.25}
$$

Notice that in a Ricci flat space, this result reduces to the expression (2.18). If one instead consider the Dirac operator coupled to the $U(1)$ connection, one gets an additional contribution in the above formula

$$\not{D}_A^\dagger \not{D}_A = -g^{\mu\nu} D_\mu D_\nu + \frac{1}{4} R + \frac{1}{2} [\gamma^i, \gamma^j] F_{ij}, \tag{2.26}$$

which is due to the curvature of the connection A,

$$F = dA = F_{ij} e^i \wedge e^j. \tag{2.27}$$

2.1.3 Dirac operator in $\mathbb{R}^3$

In this section we consider two versions of the Dirac operator in $\mathbb{R}^3$. One of them is obtained from the definition of the Dirac operator in cartesian coordinates (2.1), followed by a change of variable to spherical coordinates. The second one is obtained from the generalisation to a curved space (2.16) and the choice of the non-constant orthonormal frame of the spherical coordinates. This exercise exhibits the techniques that we use later on in the computation of Dirac operators, and it also shows that the choice of coordinates fixes the gauge of these operators.

We start by considering the metric in $\mathbb{R}^3$ in cartesian coordinates

$$ds^2 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2. \tag{2.28}$$

Using the Pauli matrices τ_i one can define a set of γ-matrices $\gamma^j = i\tau_j$, which satisfies relations (2.2) i.e. $\{\gamma^i, \gamma^j\} = -\delta_{ij}$. Thus using (2.1) we find that

$$\not{D}_{\mathbb{R}^3} = \gamma^j \partial_j = i \begin{pmatrix} \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & -\partial_3 \end{pmatrix}. \tag{2.29}$$

Observe that this operator squares to minus the Laplacian

$$\not{D}^2 = -\nabla^2 \tau_0, \tag{2.30}$$

where τ_0 is the 2×2 identity matrix. We can rewrite this operator in spherical coordinates by using a new basis of tangent vectors:

$$\hat{r} = \frac{\partial}{\partial r}, \quad \hat{\theta} = \frac{1}{r}\frac{\partial}{\partial \theta}, \quad \hat{\phi} = \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}, \tag{2.31}$$

given by the orthogonal transformation,

$$\begin{pmatrix} \hat{\theta} \\ \hat{\phi} \\ \hat{r} \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix} \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix}. \tag{2.32}$$

Denoting by T_{ij} the components of this matrix, the orthogonality property implies $T_{ik}T_{ki} = \delta_{ij}$. Using this we can recast the above operator in spherical coordinates

$$\begin{aligned} \not{D}_{\mathbb{R}^3} &= \gamma^i \partial_i = \gamma^i \delta_{ij} \partial_j = \gamma^i T_{ki} T_{kj} \partial_j, \\ &= \tau^i T_{1i} \hat{r} + \tau^i T_{2i} \hat{\theta} + \tau^i T_{3i} \hat{\phi}. \end{aligned} \tag{2.33}$$

Thus in matrix form

$$\not{D}_{\mathbb{R}^3} = \begin{pmatrix} i\cos\theta & i\sin\theta e^{-i\phi} \\ i\sin\theta e^{i\phi} & -i\cos\theta \end{pmatrix} \hat{r} + \begin{pmatrix} -i\sin\theta & i\cos\theta e^{-i\phi} \\ i\cos\theta e^{i\phi} & i\sin\theta \end{pmatrix} \hat{\theta} + \begin{pmatrix} 0 & e^{-i\phi} \\ -e^{i\phi} & 0 \end{pmatrix} \hat{\phi}. \tag{2.34}$$

A calculation shows that this operator squares to minus the Laplacian in spherical coordinates

$$-\not{D}^2_{\mathbb{R}^3} = \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2} \right] \tau_0, \tag{2.35}$$

in agreement with (2.30).

Dirac operator in $\mathbb{R}^3$ again

We now use (2.16) to compute a new version of the Dirac operator associated to the metric in $\mathbb{R}^3$ in spherical coordinates

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.36}$$

This metric can be rewritten in terms of a basis of 1-forms

$$ds^2 = (e^1)^2 + (e^2)^2 + (e^3)^2, \tag{2.37}$$

which are defined as follows

$$e^1 = r d\theta, \quad e^2 = r \sin\theta d\phi, \quad e^3 = dr. \tag{2.38}$$

The dual frame, which gives a basis of the tangent space, have the non-vanishing components

$$E_1^1 = \frac{1}{r}, \quad E_2^2 = \frac{1}{r \sin\theta}, \quad E_3^3 = 1. \tag{2.39}$$

Using the representation $\gamma^i = i\tau^i$ as before, we compute the curved γ-matrices according to (2.5),

$$\gamma^1 = \frac{1}{r}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma^2 = \frac{1}{r\sin\theta}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{2.40}$$

It remains to compute the spin connection given by the no torsion equations. So using (2.12) along with the computation

$$\begin{aligned} de^1 &= -\omega^{12} \wedge e^2 - \omega^{13} \wedge e^3, \\ de^2 &= \omega^{12} \wedge e^1 - \omega^{23} \wedge e^3, \\ de^3 &= \omega^{13} \wedge e^1 + \omega^{23} \wedge e^2, \end{aligned} \tag{2.41}$$

it follows that they satisfy the equations

$$\begin{aligned} -\omega^{12} \wedge e^2 - \omega^{13} \wedge e^3 &= -r^{-1} e^1 \wedge e^3, \\ \omega^{12} \wedge e^1 - \omega^{23} \wedge e^3 &= r^{-1} \cot\theta e^1 \wedge e^2 - r^{-1} e^2 \wedge e^3, \\ \omega^{13} \wedge e^1 + \omega^{23} \wedge e^2 &= 0. \end{aligned} \tag{2.42}$$

After a relatively straightforward calculation we find that the non-vanishing components are

$$\omega^{12} = -r^{-1} \cot\theta e^2, \quad \omega^{13} = r^{-1} e^1, \quad \omega^{23} = r^{-1} e^2. \tag{2.43}$$

One can easily check that this connection gives a vanishing curvature 2-form (2.13),

$$R^{12} = d\omega^{12} + \omega^{13} \wedge \omega^{32} = 0,$$
$$R^{13} = d\omega^{13} + \omega^{12} \wedge \omega^{23} = 0,$$
$$R^{23} = d\omega^{23} + \omega^{21} \wedge \omega^{13} = 0. \tag{2.44}$$

Using the commutation relations for the γ-matrices $\gamma^j = i\tau_j$:

$$[\gamma^i, \gamma^j] = -2\epsilon_{ijk}\gamma^k, \tag{2.45}$$

as well as (2.9), we obtain

$$\Gamma = \frac{1}{4}\epsilon_{abc}\omega^{ab}\gamma^c, \tag{2.46}$$

and a direct substitution of the solutions (2.43) gives the components

$$\Gamma_1 = -\tfrac{1}{2}\tau^2, \quad \Gamma_2 = \tfrac{1}{2}\sin\theta\tau^1 - \tfrac{1}{2}\cos\theta\tau^3, \quad \Gamma_3 = 0. \tag{2.47}$$

Using the above and the definition (2.16), we compute the Dirac operator

$$D\!\!\!/'_{\mathbb{R}^3} = \begin{pmatrix} i(\partial_r + \frac{1}{r}) & \frac{i}{r}\left(\partial_\theta - \frac{i}{\sin\theta}\partial_\phi + \frac{\cos\theta}{2\sin\theta}\right) \\ \frac{i}{r}\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi + \frac{\cos\theta}{2\sin\theta}\right) & -i(\partial_r + \frac{1}{r}) \end{pmatrix}. \tag{2.48}$$

We see that, contrary to what one could expect, this operator is different to the one in (2.34). However a lengthy calculation shows that they are actually related to each other by the $SU(2)$ gauge transformation

$$D\!\!\!/'_{\mathbb{R}^3} = u(\theta, \phi) D\!\!\!/_{\mathbb{R}^3} u^{-1}(\theta, \phi), \tag{2.49}$$

where $u(\theta, \psi) \in SU(2)$ is given by

$$u(\theta, \phi) = \begin{pmatrix} \cos\frac{\theta}{2}e^{\frac{i\phi}{2}} & \sin\frac{\theta}{2}e^{\frac{i\phi}{2}} \\ -\sin\frac{\theta}{2}e^{-\frac{i\phi}{2}} & \cos\frac{\theta}{2}e^{-\frac{i\phi}{2}} \end{pmatrix}. \tag{2.50}$$

This result shows that the choice of the coordinate system fixes the gauge of the Dirac operators.

2.2 Lens spaces and the Hopf fibration

In this section, we introduce the idea of sections of line bundles of the Hopf fibration, which will be required in the discussion of the Dirac operator on the 2-sphere. We will also need the concept of the Dirac monopole, which can be seen as the local form of a connection of the Hopf bundle. In order to generalise the structure group to $\mathbb{R}$ we identify this bundle with the Lens space $L(1, n)$.

The Hopf bundle is defined to be the $U(1)$ principal bundle over the 2-sphere, with S^3 as its total space:

$$\begin{tikzcd} U(1) \arrow[r] & S^3 \arrow[d, "\pi"] \\ & S^2 \end{tikzcd} \tag{2.51}$$

with the projection π given by the Hopf map. It will be convenient to use the usual identification of the 3-sphere; $S^3 = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} || z_1|^2 + |z_2|^2 = 1\}$ with $SU(2)$, whose elements are given by

$$h = \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}. \tag{2.52}$$

Observe that one can parametrise these in terms of Euler angles $\alpha \in [0, 2\pi), \beta \in [0, \pi]$, and $\gamma \in [0, 4\pi)$ as follows

$$z_1 = e^{-\frac{i}{2}(\gamma+\alpha)} \cos\frac{\beta}{2}, \quad z_2 = e^{\frac{i}{2}(\alpha-\gamma)} \sin\frac{\beta}{2}. \tag{2.53}$$

In this way, one can see the relation between $SU(2)$ and the generators of its Lie algebra $\mathfrak{su}(2)$:

$$t_j = -\frac{i}{2}\tau_i, \quad j = 1, 2, 3, \tag{2.54}$$

by writing

$$h = e^{\alpha t_3} e^{\beta t_2} e^{\gamma t_3} = \begin{pmatrix} e^{-\frac{i}{2}(\gamma+\alpha)} \cos\frac{\beta}{2} & -e^{\frac{i}{2}(\gamma-\alpha)} \sin\frac{\beta}{2} \\ e^{\frac{i}{2}(\alpha-\gamma)} \sin\frac{\beta}{2} & e^{\frac{i}{2}(\gamma+\alpha)} \cos\frac{\beta}{2} \end{pmatrix}. \tag{2.55}$$

With the realisation of S^3 as $SU(2)$, the elements of the fiber are $e^{\gamma t_3} \subset SU(2)$.

The group $U(1)$ has the right action on the fiber $e^{\gamma t_3}e^{\delta t_3}$ which amounts to shifting $\gamma \mapsto \gamma + \delta$. For a general element of $SU(2)$ we have

$$R(e^{i\delta}) : h \mapsto he^{\delta t_3}, \quad \delta \in [0, 4\pi). \tag{2.56}$$

The corresponding infinitesimal right action is generated by a differential operator X_3 which we will discuss in the next section.

In terms of complex coordinates (2.52), the map (2.56) reads

$$R(e^{i\delta}) : (z_1, z_2) \mapsto (z_1 e^{-i\frac{\delta}{2}}, z_2 e^{-i\frac{\delta}{2}}). \tag{2.57}$$

The ability to express the action in this way allow us to generalise the above discussion to include the Lens space $L(1,n) = S^3/\mathbb{Z}_n,\ n \neq 0$, whose generator acts via

$$h \mapsto he^{\frac{4\pi}{n}t_3}, \quad (z_1, z_2) \mapsto (z_1 e^{-i\frac{2\pi}{n}}, z_2 e^{-i\frac{2\pi}{n}}). \tag{2.58}$$

This action is as in (2.57) but with $\delta \in [0, \frac{4\pi}{n})$. As a result the associated basis of the $U(1)$ Lie algebra is $ni/2$. The vector field on $SU(2)$ generated by the $U(1)$ action is still X_3, but it is now the push-forward of the $U(1)$ generator $in/2$:

$$R_*\left(i\frac{n}{2}\right) = X_3. \tag{2.59}$$

The Hopf map can be written as a projection from $L(1,n)$ onto the unit 2-sphere inside the Lie algebra $\mathfrak{su}(2)$. Te following formula holds strictly only for S^3, but in makes sense for $L(1,n)$ too, since the image is manifestly invariant under (2.57):

$$\pi : S^3 \to S^2 \subset \mathfrak{su}(2), \quad h \mapsto ht_3h^{-1}. \tag{2.60}$$

In terms of the parametrisation (2.55),

$$\pi(h) = (\sin\beta\cos\alpha)t_1 + (\sin\beta\sin\alpha)t_2 + (\cos\beta)t_3, \tag{2.61}$$

so that our choice of Euler angles induces (β, α) as standard spherical polar coordinates on the 2-sphere.

Complex coordinates in S^2

We introduce local coordinates on S^2 by completing the above Hopf map $\pi : S^3 \to S^2$ with the stereographic projection $\mathrm{St} : S^2 \to \mathbb{C}^2$.

Writing N for the 'north pole' $(0, 0, 1) \in S^2$ and S for the 'south pole' $(0, 0, -1) \in S^2$, we define

$$U_N = S^2 \setminus \{S\}, \quad U_S = S^2 \setminus \{N\}. \tag{2.62}$$

Then, in terms the coordinates $\hat{n} = (n_1, n_2, n_3)$, stereographic projection from the south pole is

$$\mathrm{St} : U_N \subset S^2 \to \mathbb{C}^2, \quad (n_1, n_2, n_3) \mapsto z = \frac{n_1 + in_2}{1 + n_3}, \tag{2.63}$$

and stereographic projection from the north pole, followed by complex conjugation is

$$\bar{\mathrm{St}} : U_S \subset S^2 \to \mathbb{C}^2, \quad (n_1, n_2, n_3) \mapsto \zeta = \frac{n_1 - in_2}{1 - n_3}. \tag{2.64}$$

Thus $\zeta = 1/z$ and we observe that

$$z = \frac{z_2}{z_1} = \tan\frac{\beta}{2} e^{i\alpha}, \quad \zeta = \frac{z_1}{z_2} = \cot\frac{\beta}{2} e^{-i\alpha}. \tag{2.65}$$

In other words, in complex coordinates, the Hopf map followed stereographic project from the south pole is

$$\mathrm{St} \circ \pi : S^3 \to U_N, \quad (z_1, z_2) \mapsto z, \tag{2.66}$$

while the Hopf map followed by stereographic projection from the north pole and complex conjugation is

$$\bar{\mathrm{St}} \circ \pi : S^3 \to U_S, \quad (z_1, z_2) \mapsto \zeta. \tag{2.67}$$

In our discussion we also require local sections of the Hopf bundle in both complex coordinates and Euler angles. We use the same notation for both and write, on the northern patch,

$$s_N : U_N \to S^3, \quad z \mapsto \frac{1}{\sqrt{1+|z|^2}}(1,z), \qquad (\beta,\alpha) \mapsto e^{\alpha t_3} e^{\beta t_2} e^{-\alpha t_3} \tag{2.68}$$

and on the southern patch

$$s_S : U_S \to S^3, \quad \zeta \mapsto \frac{1}{\sqrt{1+|\zeta|^2}}(\zeta,1), \qquad (\beta,\alpha) \mapsto e^{\alpha t_3} e^{\beta t_2} e^{\alpha t_3}. \tag{2.69}$$

2.2.1 Forms and vector fields on $SU(2)$

The next step in our discussion of the Lens space is to describe line bundles on it. As we shall see, sections of this bundle are given by $SU(2)$ representations of vector fields generating the infinitesimal action of $SU(2)$ on itself. In the previous section we encountered one of these vector fields X_3, which generates the infinitesimal action of (2.56). In this section we review their expressions in both Euler angles and complex coordinates and discuss some of their properties.

$SU(2)$ acts on itself by right $h \mapsto he^{\delta t_i}$ and left multiplication $h \mapsto e^{-\delta t_i}h$, where the t_i are the generators of the Lie algebra defined in (2.54). We denote by X_i the generators of the infinitesimal right action

$$X_i : h \mapsto ht_i \tag{2.70}$$

and by Z_i the generators of the infinitesimal left action

$$Z_i : h \mapsto -t_i h. \tag{2.71}$$

Left invariant vector fields on S^3

Let us first consider the vector fields X_i whose explicit expressions in terms of Euler angles are

$$\begin{aligned}
X_1 &= \quad \cot\beta\cos\gamma\partial_\gamma + \sin\gamma\partial_\beta - \frac{\cos\gamma}{\sin\beta}\partial_\alpha, \\
X_2 &= -\cot\beta\sin\gamma\partial_\gamma + \cos\gamma\partial_\beta + \frac{\sin\gamma}{\sin\beta}\partial_\alpha, \\
X_3 &= \quad \partial_\gamma.
\end{aligned} \tag{2.72}$$

They satisfy the commutation relations

$$[X_i, X_j] = \epsilon_{ijk}X_k. \tag{2.73}$$

We often use the combinations

$$X_+ = X_1 + iX_2, \quad X_- = X_1 - iX_2, \tag{2.74}$$

which satisfy

$$[iX_3, X_\pm] = \pm X_\pm. \tag{2.75}$$

This implies that if F is an eigenfunction of iX_3 with eigenvalue s i.e. $iX_3F = sF$ then

$$iX_3(X_\pm F) = (X_\pm iX_3 \pm X_\pm)F = (s \pm 1)X_\pm F, \tag{2.76}$$

and thus $X_\pm$ are rising and lowering operators of the eigenvalues of iX_3. From the above we can see that

$$X_+ = ie^{-i\gamma}\left(\partial_\beta + i\frac{1}{\sin\beta}\partial_\alpha \; - i\frac{\cos\beta}{\sin\beta}\partial_\gamma\right), \quad X_- = -ie^{i\gamma}\left(\partial_\beta - i\frac{1}{\sin\beta}\partial_\alpha + i\frac{\cos\beta}{\sin\beta}\partial_\gamma\right). \tag{2.77}$$

The vector fields X_i are left-invariant and are dual to the left-invariant 1-forms σ_i on S^3 defined via

$$h^{-1}dh = \sigma_1 t_1 + \sigma_2 t_2 + \sigma_3 t_3. \tag{2.78}$$

In terms of Euler angles these forms read

$$
\begin{aligned}
\sigma_1 &= \sin\gamma d\beta - \cos\gamma \sin\beta d\alpha, \\
\sigma_2 &= \cos\gamma d\beta + \sin\gamma \sin\beta d\alpha, \\
\sigma_3 &= d\gamma + \cos\beta d\alpha.
\end{aligned} \tag{2.79}
$$

We will also require the left-invariant 1-forms in complex notation. With (2.52) we find that

$$
\sigma_1 + i\sigma_2 = 2i(z_1 dz_2 - z_2 dz_1), \quad \sigma_3 = 2i(\bar{z}_1 dz_1 + \bar{z}_2 dz_2). \tag{2.80}
$$

To compute the vector fields $X_\pm$ in complex notation we can employ

$$
t_+ = t_1 + it_2 = -i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad t_- = t_1 - it_2 = -i \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{2.81}
$$

and then by using the rule $X_\pm h \mapsto h t_\pm$ we find

$$
\begin{aligned}
X_+ &= i(z_1 \bar{\partial}_2 - z_2 \bar{\partial}_1), \\
X_- &= i(\bar{z}_2 \partial_1 - \bar{z}_1 \partial_2), \\
X_3 &= \frac{i}{2}(\bar{z}_1 \bar{\partial}_1 + \bar{z}_2 \bar{\partial}_2 - z_1 \partial_1 - z_2 \partial_2).
\end{aligned} \tag{2.82}
$$

One checks that

$$
\sigma_+(X_-) = \sigma_-(X_+) = 2, \quad \sigma_3(X_3) = 1, \tag{2.83}
$$

with all other pairings vanishing.

Right invariant vector fields on S^3

For the left-generated and right-invariant vector fields generating the infinitesimal action

$$
Z_i : h \mapsto -t_i h, \tag{2.84}
$$

we have the expressions

$$
\begin{aligned}
Z_1 &= -\frac{\cos\alpha}{\sin\beta}\partial_\gamma + \sin\alpha\partial_\beta + \cot\beta\cos\alpha\partial_\alpha, \\
Z_2 &= -\frac{\sin\alpha}{\sin\beta}\partial_\gamma - \cos\alpha\partial_\beta + \cot\beta\sin\alpha\partial_\alpha, \\
Z_3 &= -\partial_\alpha.
\end{aligned} \tag{2.85}
$$

Defining $Z_\pm = Z_1 \pm iZ_2$ and using the rule $Z_\pm : h \mapsto -t_\pm h$ we find their form in complex coordinates

$$
\begin{aligned}
Z_+ &= i(z_2\partial_1 - \bar{z}_1\bar{\partial}_2), \\
Z_- &= i(z_1\partial_2 - \bar{z}_2\bar{\partial}_1), \\
Z_3 &= \frac{i}{2}(z_1\partial_1 - z_2\partial_2 - \bar{z}_1\bar{\partial}_1 + \bar{z}_2\bar{\partial}_2).
\end{aligned} \tag{2.86}
$$

They satisfy $[Z_i, Z_j] = \epsilon_{ijk}Z_k$ (and hence $[iZ_3, Z_\pm] = \pm Z_\pm$) and commute with the right-generated vector fields X_j, $j = 1, 2, 3$.

2.2.2 $SU(2)$ representations of X_i and Z_i

As we saw in the previous section, the vector fields X_i and Z_i satisfy the relations of the Lie algebra $\mathfrak{su}(2)$ separately. They also satisfy $[X_i, Z_j] = 0$, which implies that they generate two copies of the Lie algebra $\mathfrak{su}(2)$ and commute with the Laplace operator on $SU(2)$

$$
\Delta_{S^3} = X_1^2 + X_2^2 + X_3^2 = Z_1^2 + Z_2^2 + Z_3^2. \tag{2.87}
$$

Therefore, the space of eigenfunctions of Δ_{S^3} can be decomposed into irreducible representations of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. In this section we derive these representations in complex coordinates (2.53) by fixing the eigenvalues of the commuting operators Δ_{S^3}, iX_3, iZ_3.

We use the trick of abandoning the constrain $z_1\bar{z}_1 + z_2\bar{z}_2 = 1$ and considering functions defined on all of $\mathbb{C}^2$, which belong in the kernel of the Laplace operator on $\mathbb{C}^2 \simeq \mathbb{R}^4$

$$\Box_4 = 4(\partial_1\bar{\partial}_1 + \partial_2\bar{\partial}_2). \tag{2.88}$$

To begin with, we define the differential operators on $\mathbb{C}^2$

$$D = \frac{1}{2}(z_1\partial_1 + z_2\partial_2), \quad \bar{D} = \frac{1}{2}(\bar{z}_1\bar{\partial}_1 + \bar{z}_2\bar{\partial}_2), \tag{2.89}$$

and observe that they commute with iZ_3 and are related to iX_3,

$$iX_3 = D - \bar{D}. \tag{2.90}$$

We also find the relation

$$X_+X_- = -4D\bar{D} - 2D + (|z_1|^2 + |z_2|^2)(\partial_1\bar{\partial}_1 + \partial_2\bar{\partial}_2), \tag{2.91}$$

which can be used to rewrite the Laplace operator as follows:

$$\begin{aligned} \Delta_{S^3} &= X_+X_- + (D - \bar{D}) - (D - \bar{D})^2 \\ &= -(D + \bar{D})^2 - (D + \bar{D}) + \frac{1}{4}(|z_1|^2 + |z_2|^2)\Box_4. \end{aligned} \tag{2.92}$$

We can see that an eigenfunction F of the operator

$$J = D + \bar{D}, \tag{2.93}$$

satisfying $JF = jF$, would also be an eigenfunction of the Laplace operator

$$\Delta_{S^3}F = -j(j+1)F, \tag{2.94}$$

provided $\Box_4 F = 0$. Picking $N, \bar{N} \in \frac{1}{2}\mathbb{N}_0$ and defining the functions

$$F_{N\bar{N}} = z_1^N z_2^N \bar{z}_1^{\bar{N}} \bar{z}_2^{\bar{N}}, \tag{2.95}$$

one checks that

$$DF_{N\bar{N}} = NF_{N\bar{N}}, \quad \bar{D}F_{N\bar{N}} = \bar{N}F_{N\bar{N}}, \tag{2.96}$$

and hence

$$JF_{N\bar{N}} = jF_{N\bar{N}}, \tag{2.97}$$

where $j = N + \bar{N}$. Notice that these are also eigenfunctions of iX_3:

$$iX_3 F_{N\bar{N}} = sF_{N\bar{N}}, \tag{2.98}$$

in which $s = N - \bar{N}$. As such $F_{N\bar{N}}$ are not eigenfunctions of iZ_3 as we can easily check that $iZ_3 F_{N\bar{N}} = 0$. However, we can generalise them so that they are also eigenfunctions of iZ_3, without affecting the eigenvalues of the operators iX_3, Δ_{S^3}, by putting

$$F_{NM\bar{N}\bar{M}} = z_1^{N-M} z_2^{N+M} \bar{z}_1^{\bar{N}+\bar{M}} \bar{z}_2^{\bar{N}-\bar{M}}, \tag{2.99}$$

where $-N \leq M \leq N$ and $-\bar{N} \leq \bar{M} \leq \bar{N}$. In this way we have

$$iZ_3 F_{NM\bar{N}\bar{M}} = mF_{NM\bar{N}\bar{M}}, \tag{2.100}$$

where $m = M + \bar{M}$. Observe that we can recast the $F_{NM\bar{N}\bar{M}}$ as

$$F_{NM\bar{N}\bar{M}} = z_1^{s-m+k} z_2^{j+m-k} \bar{z}_1^{k} \bar{z}_2^{j-s-k}, \tag{2.101}$$

where we have defined $k = \bar{M} + \bar{N}$. These functions do not satisfy the condition $\Box_4 F_{NM\bar{N}\bar{M}} = 0$ in general. However, we can construct linear combinations of them Y^j_{sm} satisfying $\Box_4 Y^j_{sm} = 0$ by fixing N and $\bar{N}$, and hence also j. By amending discussion in [20] we find that the following functions satisfy this condition

$$Y^j_{sm} = c^j_{sm} \sum_k \frac{(j+m)!}{(j+m-k)!k!} \frac{(j-m)!(-1)^{j-s-k}}{(j-s-k)!(s-m+k)!} z_1^{s-m+k} z_2^{j+m-k} \bar{z}_1^{k} \bar{z}_2^{j-s-k}, \tag{2.102}$$

where

$$c^j_{sm} = \left[\frac{(j+s)!(j-s)!}{(j+m)!(j-m)!} \right]^{1/2}, \tag{2.103}$$

where the parameter k runs over the values so that the powers of the complex coordinates are positive. By construction, they satisfy

$$\Delta_{S^3} Y^j_{sm} = j(j+1) Y^j_{sm}, \quad iX_3 Y^j_{sm} = sY^j_{sm}, \quad iZ_3 Y^j_{sm} = mZ^j_{sm}. \tag{2.104}$$

It also follows that

$$X_+Y^j_{sm} = -i[(j-s)(j+s+1)]^{1/2}Y^j_{s+1m}, \quad X_-Y^j_{sm} = -i[(j+s)(j-s+1)]^{1/2}Y^j_{s-1m}, \tag{2.105}$$

and so $X_\pm$ are raising and lowering operators of the eigenvalue s of iX_3, as expected.

For fixed j, both s and m take values in the interval $-j, -j+1, \ldots, j-1, j$, and so the space of the polynomials Y^j_{sm} has dimension $(2j+1)(2j+1)$. If s is fixed too, we obtain an $SU(2)$ representation of dimension $2j+1$. There are two special cases that we will encounter later. The first one is obtained by setting $s = j$. Then $k = 0$ and we obtain the holomorphic basis

$$c^j_{jm} z_1^{j-m} z_2^{j+m}, \quad m = -j, -j+1, \ldots, j-1, j. \tag{2.106}$$

The second one is obtained by setting $s = -j$. Then $k = j+m$ and we obtain the antiholomorphic basis

$$c^j_{-jm} \bar{z}_1^{j+m}(-\bar{z}_2^{j-m}), \qquad m = -j, -j+1, \ldots, j-1, j. \tag{2.107}$$

2.2.3 Associated lined bundles

Having defined simultaneous eigenfunctions Y^j_{sm} of the operators Δ_{S^3}, iX_3 and iZ_3, we now show that they are sections of line bundles associated to the Lens spaces and that we can obtain local sections on S^2 via pull-back with (2.68) and (2.69).

We can describe sections of line bundles associated to the Lens spaces in terms of equivariant functions on the total space

$$F : L(1,n) \to \mathbb{C}, \tag{2.108}$$

that satisfy

$$F(he^{\delta t_3}) = e^{-i\frac{n}{2}\delta} F(h), \quad \delta \in \left[0, \frac{4\pi}{n}\right], \tag{2.109}$$

in which h represents an equivalent class of the Lens space. Because X_3 is the generator of the infinitesimal right action, the infinitesimal form of the equivariant condition can be expressed as

$$iX_3F = i\left(\frac{d}{d\delta}e^{-i\frac{n}{2}\delta}F\right)\Big|_{\delta=0} = \frac{n}{2}F. \tag{2.110}$$

From the previous analysis $F(he^{\delta t_3}) = F(e^{-i\delta/2}z_1, e^{-i\delta/2}z_2)$, and so writing $\lambda = e^{-i\delta/2}$ we have

$$F(\lambda z_1, \lambda z_2) = \lambda^n F(z_1, z_2). \tag{2.111}$$

We can obtain local sections on the northern and southern patches U_N and U_S via pull-back with the local sections (2.68) and (2.69):

$$f_N = s_N^* F, \quad f_S = s_S^* F. \tag{2.112}$$

So explicitly

$$f_N(z) = F\left(\frac{1}{\sqrt{q}}(1, z)\right), \quad f_S(z) = F\left(\sqrt{\frac{\bar{z}}{z}}\frac{1}{\sqrt{q}}(1, z)\right), \tag{2.113}$$

and then using (2.111) we get the patching condition

$$f_S = e^{-in\alpha}f_N = \left(\frac{\bar{z}}{z}\right)^{\frac{n}{2}} f_N. \tag{2.114}$$

The line bundle associated to $L(1, n)$ whose sections satisfy the previous condition is often denoted as H^n, the nth tensor power of the hyperplane bundle H. The latter is the dual bundle of the tautological line bundle L over $\mathbb{CP}^1$ whose fibre over a point $\ell \in \mathbb{CP}^1$ is the line in $\mathbb{C}^2$ defined by ℓ:

$$L = \{(\ell, (w_1, w_2)) \subset \mathbb{CP}^1 \times \mathbb{C}^2 | (w_1, w_2) \in \ell\}. \tag{2.115}$$

For the hyperplane bundle H over $\mathbb{CP}^1$, the fibre over a point $\ell \in \mathbb{CP}^1$ is the dual space ℓ^*.

We observe that the functions Y^j_{sm} (2.102) satisfy the equivariant condition

(2.111) with $n = 2s$:

$$Y^j_{sm}(\lambda z_1, \lambda z_2) = \lambda^{2s} Y^j_{sm}. \tag{2.116}$$

Notice from (2.104) that they also satisfy the infinitesimal version of the equivariant condition (2.110), and so they are sections of H^{2s}.

We can obtain for example local sections of Y^j_{sm} on the north pole via pull-back with (2.68) which amounts to replacing $z_1 \to \frac{1}{\sqrt{q}}$, $z_2 \to \frac{z}{\sqrt{q}}$ in (2.102):

$$Y^j_{sm} = c^j_{sm} \sum_k \frac{(j+m)!}{(j+m-k)!k!} \frac{(j-m)!(-1)^{j-s-k}}{(j-s-k)!(s-m+k)!} q^{-j} z^{j+m-k} \bar{z}^{j-s-k}. \tag{2.117}$$

Here Y^j_{sm} is actually $s^*_N Y^j_{sm}$ but we use the same notation. These sections will appear in the discussion of Dirac operators on the 2-sphere.

2.2.4 The Dirac monopole

We end our discussion on the Lens space by describing the Dirac monopole as the local form of a rotationally invariant connection on the Lens space.

Using (2.59) we see that the requirement for a 1-form $\mathcal{A}$ to be a connection 1-form on $L(1, n)$ is

$$\mathcal{A}(X_3) = \frac{in}{2}, \tag{2.118}$$

while 'rotationally invariant' means invariant under the left-action of $SU(2)$ on $L(1, n)$. Since σ_3 (2.79) is the dual of X_3, we see that

$$\mathcal{A} = \frac{in}{2}\sigma_3 = \frac{in}{2}(d\gamma + \cos\beta d\alpha), \tag{2.119}$$

satisfies these requirements. Its curvature

$$F = d\mathcal{A} = -\frac{in}{2}\sin\beta d\beta \wedge d\alpha, \tag{2.120}$$

gives the magnetic field of the Dirac magnetic monopole. This field strength is called magnetic since it only has magnetic components.

The pull-back of the potential $\mathcal{A}$ with the local sections (2.68) and (2.69) gives local gauge potentials for the Dirac monopole on the north and south poles:

$$s_N^*\mathcal{A} = A_N^n = \frac{in}{2}(-1+\cos\beta)d\alpha, \quad s_S^*\mathcal{A} = A_S^n = \frac{in}{2}(1+\cos\beta)d\alpha. \tag{2.121}$$

These potentials are related by the $U(1)$ transformation given by the transition function $g_{SN}(\alpha) = e^{-in\alpha}$, which is defined on the intersection $U_N \cap U_S$,

$$A_S^n = A_N^n + g_{SN} d g_{SN}^{-1}, \tag{2.122}$$

and satisfy $F = dA_N^n = dA_S^n$. The charge n has to be an integer so that the transition function is single-valued. Now we compute the the first Chern number C_1 of the $U(1)$ fiber bundle, which is given by the integral [17] of the Chern form

$$c_1 = \frac{i}{2\pi}\mathrm{Tr}(F). \tag{2.123}$$

Using this we see that the charge is equal to the only non-trivial Chern number of the bundle

$$C_1 = \int_{S^2} c_1 = n. \tag{2.124}$$

Since the potential A_N^n is well defined on U_N, we rewrite it in terms of z and q as

$$A_N^n = \frac{n}{2q}(z d\bar{z} - \bar{z} dz). \tag{2.125}$$

Similarly, on U_S, we have

$$A_S^n = \frac{n}{2}\frac{\zeta d\bar{\zeta} - \bar{\zeta} d\zeta}{1+|\zeta|^2}. \tag{2.126}$$

For the curvature we find

$$F = n(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2) = n\frac{dz \wedge d\bar{z}}{(1+|z|^2)^2} = n\frac{d\zeta \wedge d\bar{\zeta}}{(1+|\zeta|^2)^2}, \tag{2.127}$$

with the equalities holding wherever the expressions are defined.

Chapter 3

Taub-NUT Zero-Modes

3.1 Dirac operator on the 2-sphere

3.1.1 Twisted Dirac operator

In this section we consider the Dirac operator on the 2-sphere coupled to the Dirac monopole. We show that this operator acts on sections of the hyperplane H^n discussed in Chapter 2.

The computation of the Dirac operator on the sphere works in exactly the same way as in $\mathbb{R}^3$. In fact we have already done this calculation implicitly when computing the Dirac operator on $\mathbb{R}^3$ in Sect. 2.1.3. One can see this from the metric in $\mathbb{R}^3$ (2.36), which replacing $(\theta, \phi) \mapsto (\beta, \alpha)$, reduces to the metric on the unit sphere by setting $r = 1$:

$$ds^2 = d\beta^2 + \sin^2 \beta d\alpha^2. \tag{3.1}$$

This metric admits a 2-bein

$$e^1 = d\beta, \quad e^2 = \sin \beta d\alpha, \tag{3.2}$$

with the dual vector fields

$$\tilde{E}_1 = \partial_\beta, \quad \tilde{E}_2 = \frac{1}{\sin \beta} \partial_\alpha. \tag{3.3}$$

Following a similar procedure as in Sect. 2.1.3, we find that the Dirac operator on the sphere is given by the off-diagonal components of the Dirac operator in $\not{D}'_{\mathbb{R}^3}$ (2.48). More precisely, we have

$$\not{D}'_{\mathbb{R}^3} = i \begin{pmatrix} (\partial_r + \frac{1}{r}) & 0 \\ 0 & -(\partial_r + \frac{1}{r}) \end{pmatrix} + \frac{1}{r}\not{D}_{S^2}. \tag{3.4}$$

The ability to write the operator in this way will be useful to compute its zero-modes by using separation of variables. Before doing that, we will first focus on the zero modes of $\not{D}_{S^2}$. In order to make our discussion compatible with the previous analysis of sections of the hyperplane bundle, we will use the complex coordinate z on the north pole of of the 2-sphere (2.65), rather than spherical angles.

First we observe that, in terms of z, the metric (3.1) has the form

$$ds^2 = \frac{4}{q^2} dz d\bar{z}, \tag{3.5}$$

where

$$q = 1 + z\bar{z}. \tag{3.6}$$

Then writing $z = y_1 + iy_2$ we see that the metric admits the 2-bein

$$e^1 = \frac{2}{q} dy_1, \quad e^2 = \frac{2}{q} dy_2, \tag{3.7}$$

with dual vector fields

$$E_1 = \frac{q}{2}\frac{\partial}{\partial y_1}, \quad E_2 = \frac{q}{2}\frac{\partial}{\partial y_2}. \tag{3.8}$$

One can check that the two frames are related by a rotation:

$$\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \tilde{E}_1 \\ \tilde{E}_2 \end{pmatrix}. \tag{3.9}$$

This rotation leads to a gauge change for the associated spin bundles which we will encounter later in our discussion.

As before we use the representation

$$\gamma^1 = i\tau^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma^2 = i\tau^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3.10}$$

and find the non-vanishing component of the spin connection (2.8),

$$\omega^{12} = y^1 e^2 - y^2 e^1 = \frac{2}{q}(y^1 dy^2 - y^2 dy^1). \tag{3.11}$$

Using this we compute the spin connection 1-form (2.9)

$$\Gamma_1 = \begin{pmatrix} -\frac{i}{q}y^2 & 0 \\ 0 & \frac{i}{q}y^2 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} \frac{i}{q}y^1 & 0 \\ 0 & -\frac{i}{q}y^1 \end{pmatrix}. \tag{3.12}$$

With these ingredients we can now compute the Dirac operator (2.16),

$$\not{D}_{S^2} = \begin{pmatrix} 0 & i(q\partial_z - \frac{1}{2}\bar{z}) \\ i(q\bar{\partial}_z - \frac{1}{2}z) & 0 \end{pmatrix}, \tag{3.13}$$

where we have defined

$$\partial_z = \frac{1}{2}\left(\frac{\partial}{\partial y^1} - i\frac{\partial}{\partial y^2}\right), \quad \bar{\partial}_z = \frac{1}{2}\left(\frac{\partial}{\partial y^1} + i\frac{\partial}{\partial y^2}\right). \tag{3.14}$$

The operator $\not{D}_{S^2}$ does not have normalisable zero-modes. However one can obtain zero-modes by coupling the operator to the gauge potential of the Dirac monopole which, on the north pole, has the local form (2.125). Thus the twisted version is obtained by replacing

$$\partial_z \to \partial_z - \frac{n}{2q}\bar{z}, \quad \bar{\partial}_z \to \bar{\partial}_z + \frac{n}{2q}z. \tag{3.15}$$

Doing this we obtain

$$\not{D}_{S^2,n} = \begin{pmatrix} 0 & i\partial^{\downarrow}_{\tilde{s}} \\ i\partial^{\uparrow}_{s} & 0 \end{pmatrix}, \tag{3.16}$$

in which

$$\partial^{\uparrow}_{s} = q\bar{\partial}_z + sz, \quad \partial^{\downarrow}_{\tilde{s}} = q\partial_z - \tilde{s}\bar{z}, \tag{3.17}$$

where we have defined

$$s = \frac{1}{2}(n-1), \quad \tilde{s} = \frac{1}{2}(n+1). \tag{3.18}$$

As discussed earlier, the functions (2.102) are sections of powers of the hyperplane bundle H, and we can obtain local representations of them by pull-back with (2.68) and (2.69). In particular, local sections on the north pole are given by (2.117). We now consider the action of the $\not{D}_{S^2,n}$ on these sections. First we observe that the operators (3.17) admit the factorisation

$$\partial_s^{\uparrow} = q^{-s+1}\bar{\partial}_z q^s, \quad \partial_{\tilde{s}}^{\downarrow} = q^{\tilde{s}+1}\partial_z q^{-\tilde{s}}. \tag{3.19}$$

Using this, one can compute for example the action of $\partial_s^{\uparrow}$ on the base functions of the local sections (2.117) of H^{2s}

$$q^{-j} z^{j+m-k} \bar{z}^{j-s-k}, \tag{3.20}$$

as follows

$$\begin{aligned}\partial_s^{\uparrow} q^{-j} z^{j+m-k} \bar{z}^{j-s-k} &= (s-j) q^{-j} z^{j+m-k+1} \bar{z}^{j-s-k} + (j-s-k) q^{-j+1} z^{j+m-k} \bar{z}^{j-s-k-1} \\ &= -k q^{-j} z^{j+m-k+1} \bar{z}^{j-s-k} + (j-s-k) q^{-j} z^{j+m-k} \bar{z}^{j-s-k-1}.\end{aligned} \tag{3.21}$$

Then we deduce the relation

$$\partial_s^{\uparrow} Y^j_{sm} = -[(j+s+1)(j-s)]^{\frac{1}{2}} Y^j_{s+1m}, \tag{3.22}$$

which implies that $\partial_s^{\uparrow} : C^{\infty}(H^{2s}) \to C^{\infty}(H^{2(s+1)})$, and so this operator increases the power of the hyperplane bundle in 2. Similarly, using

$$\partial_{\tilde{s}}^{\downarrow} q^{-j} z^{j+m-k} \bar{z}^{j-\tilde{s}-k} = -(\tilde{s}-m+k) q^{-j} z^{j+m-k} \bar{z}^{j-\tilde{s}+1-k} + (j+m-k) q^{-j} z^{j+m-k-1} \bar{z}^{j-\tilde{s}-k}, \tag{3.23}$$

it follows that

$$\partial_{\tilde{s}}^{\downarrow} Y^j_{\tilde{s}m} = [(j-\tilde{s}+1)(j+\tilde{s})]^{\frac{1}{2}} Y^j_{\tilde{s}-1m}, \tag{3.24}$$

which implies $\partial^{\downarrow}_{\tilde{s}} : C^\infty(H^{2\tilde{s}}) \to C^\infty(H^{2(\tilde{s}-1)})$. We can recast these results in terms of the parameter n (3.18) as

$$\begin{aligned} \partial^{\uparrow}_{s} &: C^\infty(H^{n-1}) \to C^\infty(H^{n+1}), \\ \partial^{\downarrow}_{\tilde{s}} &: C^\infty(H^{n+1}) \to C^\infty(H^{n-1}), \end{aligned} \tag{3.25}$$

and so the Dirac operator is a map

$$\not{D}_{S^2,n} : C^\infty(H^{n-1} \oplus H^{n+1}) \to C^\infty(H^{n-1} \oplus H^{n+1}). \tag{3.26}$$

3.1.2 The Edth operators ð, $\bar{\text{ð}}$

In many papers dealing with the Dirac operator on the 2-sphere, calculations are carried out in terms of spherical coordinates. In particular, eigenfunctions like the spin spherical harmonics are written as functions of the angles β and α. In order to facilitate comparisons between our discussion and treatments involving spherical coordinates, we note that in spherical coordinates

$$\begin{aligned} \partial^{\uparrow}_{s} &= e^{i\alpha}\left(\partial_\beta + i\frac{1}{\sin\beta}\partial_\alpha + s\tan\frac{\beta}{2}\right), \\ \partial^{\downarrow}_{\tilde{s}} &= e^{-i\alpha}\left(\partial_\beta - i\frac{1}{\sin\beta}\partial_\alpha - \tilde{s}\tan\frac{\beta}{2}\right). \end{aligned} \tag{3.27}$$

It is now easy to establish a link with the "edth" operators which were first introduced by Penrose and Newman [21] and which are frequently used to write the Dirac operator on S^2

$$\eth_s = \partial_\beta + i\frac{1}{\sin\beta}\partial_\alpha - s\frac{\cos\beta}{\sin\beta}, \qquad \bar{\eth}_{\tilde{s}} = \partial_\beta - i\frac{1}{\sin\beta}\partial_\alpha + \tilde{s}\frac{\cos\beta}{\sin\beta}. \tag{3.28}$$

We observe that

$$\partial^{\uparrow}_{s} e^{is\alpha} = e^{i(s+1)}\eth_s \quad \text{and} \quad \partial^{\downarrow}_{\tilde{s}} e^{i\tilde{s}\alpha} = e^{i(\tilde{s}-1)\alpha}\bar{\eth}_{\tilde{s}}. \tag{3.29}$$

They reflect the gauge change from complex to spherical coordinates (3.9).

In order to relate the discussion here to that of the Dirac operator on TN later in this work we need to understand how $\eth_s^{\uparrow}$ and $\eth_{\tilde{s}}^{\downarrow}$ are related to the vector fields X_1, X_2, X_3 of the infinitesimal $SU(2)$ right action on itself (2.70). Earlier we showed that $X_\pm = X_1 \pm iX_2$ are the rising and lowering operators of the parameter s, which is the eigenvalue of iX_3 when acting on sections (2.102) of H^{2s}. Since $\eth_s^{\uparrow}$ and $\eth_{\tilde{s}}^{\downarrow}$ also raise (3.22) and lower (3.24) the value of this parameter when acting on the local sections (2.117), we expect the former to be related to X_+ and the later to X_-. This relation was first noticed using different notation and convention from ours, in [22]. We now exhibit it in our notation.

Consider a section of H^{2s} (or H^{n-1}) in its equivariant form (2.108) as a function F of the complex variables z_1, z_2 satisfying the constraint (2.109). We denote pull-back with the local section (2.68) s_N^*, so that in particular

$$(s_N^*(X_+F))(z) = i\left(z_1\bar{\partial}_2 F - z_2\bar{\partial}_1 F\right)\Big|_{z_1=\frac{1}{\sqrt{q}}, z_2=\frac{z}{\sqrt{q}}} . \tag{3.30}$$

Then we evaluate

$$\begin{aligned} i\eth_s^{\uparrow}(s_N^*F)(z) &= i(q\bar{\partial}_z + sz)F\left(\frac{1}{\sqrt{q}}, \frac{z}{\sqrt{q}}\right), \\ &= iq\bar{\partial}_z F\left(\frac{1}{\sqrt{q}}, \frac{z}{\sqrt{q}}\right) - z(X_3F(z_1,z_2))\Big|_{z_1=\frac{1}{\sqrt{q}}, z_2=\frac{z}{\sqrt{q}}} , \end{aligned} \tag{3.31}$$

where we have used the constrain (2.108). The first term can be computed directly

$$\begin{aligned} &iq\bar{\partial}_z F\left(\frac{1}{\sqrt{q}}, \frac{z}{\sqrt{q}}\right) = \\ &= iq\left[\frac{d}{d\bar{z}}\left(\frac{1}{\sqrt{q}}\right)\partial_1 + \frac{d}{d\bar{z}}\left(\frac{z}{\sqrt{q}}\right)\partial_2 + \frac{d}{d\bar{z}}\left(\frac{1}{\sqrt{q}}\right)\bar{\partial}_1 + \frac{d}{d\bar{z}}\left(\frac{\bar{z}}{\sqrt{q}}\right)\bar{\partial}_2\right] F(z_1,z_2)\Big|_{z_1=\frac{1}{\sqrt{q}}, z_2=\frac{z}{\sqrt{q}}} \\ &= \left[-\frac{iz}{2\sqrt{q}}\partial_1 - \frac{iz^2}{2\sqrt{q}}\partial_2 - \frac{iz}{2\sqrt{q}}\bar{\partial}_1 + \left(\frac{i}{\sqrt{q}} + \frac{iz\bar{z}}{2\sqrt{q}}\right)\bar{\partial}_2\right] F(z_1,z_2)\Big|_{z_1=\frac{1}{\sqrt{q}}, z_2=\frac{z}{\sqrt{q}}} , \end{aligned} \tag{3.32}$$

Using the expression for X_3, given in (2.82), we find for the second term

$$\begin{aligned}
-z(X_3F(z_1,z_2))\Big|_{z_1=\frac{1}{\sqrt{q}},z_2=\frac{z}{\sqrt{q}}} &= \\
&-\frac{iz}{2}\left[(\bar{z}_1\bar{\partial}_1+\bar{z}_2\bar{\partial}_2-z_1\partial_1-z_2\partial_2)F(z_1,z_2)\right]\Big|_{z_1=\frac{1}{\sqrt{q}},z_2=\frac{z}{\sqrt{q}}} \\
&=\left(-\frac{iz}{2\sqrt{q}}\bar{\partial}_1-\frac{iz\bar{z}}{2\sqrt{q}}\bar{\partial}_2+\frac{iz}{2\sqrt{q}}\partial_1+\frac{iz^2}{2\sqrt{q}}\partial_2\right)F(z_1,z_2)\Big|_{z_1=\frac{1}{\sqrt{q}},z_2=\frac{z}{\sqrt{q}}}\,.
\end{aligned} \tag{3.33}$$

Then we see that

$$i\partial_s^{\uparrow}(s_N^*F)(z)=(s_N^*(X_+F))(z), \tag{3.34}$$

in agreement with relations (2.105) and (3.22) for the sections Y^j_{sm} of H^{2s}. Thus, the operator $\partial_s^{\uparrow}$ acting 'downstairs' on a local section is the pull-back of the $SU(2)$ raising operator X_+ acting upstairs on equivariant functions. Similarly, considering sections of $H^{2\tilde{s}}$ (or H^{n+1}) one finds that $\partial_{\tilde{s}}^{\downarrow}$ is related to X_- via

$$-i\partial_{\tilde{s}}^{\downarrow}(s_N^*F)(z)=(s_N^*(X_-F))(z), \tag{3.35}$$

where now, the equivariant condition reads $iX_3F=\tilde{s}F$. Again we have an agreement with relations (2.105) and (3.24) for sections $Y^j_{\tilde{s}m}$ of $H^{2\tilde{s}}$.

Using the above results along with the notation

$$C^\infty(S^3,\mathbb{C})_s=\{F:S^3\to\mathbb{C}\,|iX_3F=sF\} \tag{3.36}$$

for the space of sections H^{n-1} in the equivariant form, we define an equivalent operator to $\not{D}_{S^2,n}$ acting 'upstairs' as

$$\not{D}^*_{S^2,n}=\begin{pmatrix}0 & X_-\\ -X_+ & 0\end{pmatrix}:C^\infty(S^3,\mathbb{C})_s\oplus C^\infty(S^3,\mathbb{C})_{\tilde{s}}\to C^\infty(S^3,\mathbb{C})_s\oplus C^\infty(S^3,\mathbb{C})_{\tilde{s}}, \tag{3.37}$$

with s and $\tilde{s}$ defined in (3.18). This operator commutes with the operator

$$\hat{n}=2iX_3+\tau_3:C^\infty(S^3,\mathbb{C})_s\oplus C^\infty(S^3,\mathbb{C})_{\tilde{s}}\to C^\infty(S^3,\mathbb{C})_s\oplus C^\infty(S^3,\mathbb{C})_{\tilde{s}}, \tag{3.38}$$

which we interpret as 'Chern-number operator' since it acts as a multiple of the identity with eigenvalue $2s+1 = 2\tilde{s}-1 = n$. We will encounter it in a slightly modified form in our discussion of the Dirac operator on the TN space.

3.1.3 Zero-modes on the 2-sphere

We now have all the tools we need to compute the zero-modes of the operator $\not{D}_{S^2,n}$. Working in the patch U_N we write a spinor there as

$$\psi^N = \begin{pmatrix} f_1^N \\ f_2^N \end{pmatrix}, \tag{3.39}$$

in which f_1^N is a local section of H^{n-1} and f_2^N a local section of H^{n+1}. Then

$$\not{D}_{S^2,n}\psi^N = 0 \;\Leftrightarrow\; \partial_s^\uparrow f_1^N = 0, \qquad \partial_{\tilde{s}}^\downarrow f_2^N = 0. \tag{3.40}$$

Using the factorisation (3.19) we deduce that solutions are of the form

$$f_1^N(z) = \frac{1}{q^s} p_1(z), \qquad f_2^N(z) = q^{\tilde{s}} p_2(\bar{z}), \tag{3.41}$$

where the functions $p_1(z)$ and $p_2(\bar{z})$ are a priori holomorphic and anti-holomorphic polynomials respectively. In order for $f_1^N(z)$ to be a local section of H^{n-1}, it has to satisfy the patching condition (2.114),

$$f_1^S(z) = \frac{1}{q^s}\left(\frac{\bar{z}}{z}\right)^s p_1(z). \tag{3.42}$$

To determine the degree of $p_1(z)$, we transform to $\zeta = 1/z$ and find

$$f_1^S\left(\frac{1}{\zeta}\right) = \frac{\zeta^{2s}}{(1+\zeta\bar{\zeta})^s} p_1\left(\frac{1}{\zeta}\right). \tag{3.43}$$

For this to be well-defined at $z \mapsto \infty$ or $\zeta = 0$, the degree of p_1 has to be $\leq 2s = n-1$. Thus in this case n has to be an integer ≥ 1. The dimension of the space of zero-modes is then $2s+1 = n$.

In the same way, we have to check that the second component

$$f_2^S(z) = q^{\tilde{s}} \left(\frac{\bar{z}}{z}\right)^{\tilde{s}} p_2(\bar{z}), \tag{3.44}$$

is well-defined at $z = \infty$. So we transform to ζ and find

$$f_2^S\left(\frac{1}{\zeta}\right) = \frac{(1+\zeta\bar{\zeta})^{\tilde{s}}}{\bar{\zeta}^{2\tilde{s}}} p_2\left(\frac{1}{\bar{\zeta}}\right), \tag{3.45}$$

which restrict p_2 to be a polynomial of degree $\leq -2\tilde{s} = -n+1$. So in this case n has to be an integer ≤ -1. The dimension of the space of zero-modes is $-2\tilde{s}+1 = -n$.

Since $p_1(z)$ is a polynomial of degree $n-1$ we may write $p_1(z) = \sum_{k=0}^{n-1} a_k z^k,\ \ n \geq 1$. Then we see that the north component f_1^N can be written as the pull-back with the local section (2.68) of a homogeneous polynomial in two complex variables

$$f_1^N(z) = q^{\frac{-n+1}{2}} p_1(z) = s_N^* \left(\sum_{k=0}^{n-1} a_k z_1^{n-1-k} z_2^k\right), \quad n \geq 1. \tag{3.46}$$

Or replacing the index k by $m = k - j$, where $j = \frac{n-1}{2}$,

$$f_1^N(z) = s_N^* \left(\sum_{m=-j}^{j} a_m z_1^{j-m} z_2^{j+m}\right). \tag{3.47}$$

This shows that $f_1^N(z)$ can be viewed as the local expression of a linear combination of the holomorphic basis (2.106). In the same way, writing $p_2(\bar{z}) = \sum_{k=0}^{-n-1} \alpha_k \bar{z}^k,\ \ n \leq -1$, we see that

$$f_2^N = q^{\frac{n+1}{2}} p_2(\bar{z}) = s_N^* \left(\sum_{k=0}^{-n-1} \alpha_k \bar{z}_1^{-n-1-k} \bar{z}_2^k\right), \quad n \leq -1. \tag{3.48}$$

Similarly we can recast the sum in terms of $m = j - k$

$$f_2^N = s_N^* \left(\sum_{m=-j}^{j} \alpha_m \bar{z}_1^{j+m} \bar{z}_2^{j-m}\right), \tag{3.49}$$

where now $j = \frac{-n-1}{2}$. An so $f_2^N(\bar{z})$ can be viewed as the pull-back of a linear combination of the anti-holomorphic basis (2.107). This viewpoint is helpful in understanding the $SU(2)$ action on the zero-modes, and also provides a link with the zero-modes of the TN space of the next chapter.

Summing up, the zero modes of $\not{D}_{S^2,n}$ take the following form on U_N:

$$\psi^N(z) = \begin{pmatrix} q^{\frac{1}{2}(1-n)} \sum_{k=0}^{n-1} a_k z^k \\ 0 \end{pmatrix} \text{ if } n \geq 1, \quad \psi^N(\bar{z}) = \begin{pmatrix} 0 \\ q^{\frac{1}{2}(1+n)} \sum_{k=0}^{-n-1} \alpha_k \bar{z}^k \end{pmatrix} \text{ if } n \leq -1. \tag{3.50}$$

3.1.4 Zero-modes as irreducible $SU(2)$ representations

The $|n|$-dimensional space of zero-modes of $\not{D}_{S^2,n}$ is naturally acted on by the double cover $SU(2)$ of the isometry group of the 2-sphere. The quickest way to see that the space of zero modes is actually the n-dimensional irreducible representation of $SU(2)$ is to use the description of the zero modes as homogeneous polynomials in the two complex variables z_1, z_2 in (3.46) and (3.48). As we saw earlier, polynomials of the basis (2.106), (2.107) span the irreducible $SU(2)$ representations of dimension n for $n > 0$ and $-n$ for $n < 0$.

Explicitly, an $SU(2)$ element

$$U = \begin{pmatrix} b & \bar{a} \\ -a & \bar{b} \end{pmatrix}, \qquad |a|^2 + |b|^2 = 1, \tag{3.51}$$

acts on the polynomials (3.46) and (3.48) via pull-back with the inverse

$$U^{-1} = \begin{pmatrix} \bar{b} & -\bar{a} \\ a & b \end{pmatrix}, \tag{3.52}$$

i.e. by mapping the arguments (z_1, z_2) according to

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} \bar{b} & -\bar{a} \\ a & b \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \bar{b}z_1 - \bar{a}z_2 \\ az_1 + bz_2 \end{pmatrix}, \tag{3.53}$$

and $(\bar{z}_1, \bar{z}_2)$ analogously.

The transformation of the zero-modes (3.50) is induced by pulling back the action (3.53). The non-trivial nature of the line bundles implies an additional phase factor or multiplier, as we shall now show. We introduce the notation u^{-1} for the mapping induced by (3.53) on the quotient $z = z_2/z_1$:

$$u^{-1} : z \mapsto \frac{a + bz}{\bar{b} - \bar{a}z}. \tag{3.54}$$

Exploiting $|a|^2 + |b|^2 = 1$ the function q (3.6) satisfies

$$\begin{aligned} q(u^{-1}(z)) &= 1 + \left(\frac{a + bz}{\bar{b} - \bar{a}z}\right)\left(\frac{\bar{a} + \bar{b}\bar{z}}{b - a\bar{z}}\right) \\ &= \frac{q(z)}{(\bar{b} - \bar{a}z)(b - a\bar{z})}. \end{aligned} \tag{3.55}$$

We can generalise this action to a local section $f : U_N \to \mathbb{C}$ which is the pull-back of a function $F : S^3 \to \mathbb{C}$ satisfying the condition (2.111), as follows

$$\rho_s(U)f = s_N^*(F \circ U^{-1}). \tag{3.56}$$

Using (2.111) and (3.55) we find

$$\begin{aligned} \rho_s(U)f &= s_N^* F(\bar{b}z_1 - \bar{a}z_2, az_1 + bz_2) \\ &= F\left(\frac{\bar{b} - \bar{a}z}{\sqrt{q}}, \frac{a + bz}{\sqrt{q}}\right) \\ &= F\left(\frac{\sqrt{\bar{b} - \bar{a}z}}{\sqrt{b - a\bar{z}}} \frac{1}{\sqrt{\frac{q}{(\bar{b} - \bar{a}z)(b - a\bar{z})}}}, \frac{\sqrt{\bar{b} - \bar{a}z}}{\sqrt{b - a\bar{z}}} \frac{\frac{a+bz}{\bar{b}-\bar{a}z}}{\sqrt{\frac{q}{(b-a\bar{z})(\bar{b}-a\bar{z})}}}\right) \\ &= \mu_s(U; z)F(u^{-1}(z)), \end{aligned} \tag{3.57}$$

where the multiplier μ_s is

$$\mu_s(U; z) = \left(\frac{\bar{b} - \bar{a}z}{b - a\bar{z}}\right)^s. \tag{3.58}$$

We need to check this is actually an action i.e. that this satisfy the relation,

$$
\begin{aligned}
\rho_s(UV)f &= \rho_s(U)[\rho_s(V)f(z)] \\
&= \mu_s(U;z)[\rho_s(V)f(u^{-1}z)] \\
&= \mu_s(U;z)\mu_s(V;u^{-1}z)f(v^{-1}u^{-1}z),
\end{aligned} \tag{3.59}
$$

where V is also element of $SU(2)$

$$
U = \begin{pmatrix} d & \bar{c} \\ -c & \bar{d} \end{pmatrix}, \qquad |c|^2 + |d|^2 = 1. \tag{3.60}
$$

This amounts to check that

$$
\mu_s(UV;z) = \mu_s(U;z)\mu_s(V;u^{-1}z). \tag{3.61}
$$

Indeed

$$
\begin{aligned}
\mu_s(U;z)\mu_s(V;u^{-1}z) &= \left(\frac{\bar{b}-\bar{a}z}{b-a\bar{z}}\right)^s \left(\frac{\bar{d}-\bar{c}\left(\frac{a+bz}{\bar{b}-\bar{a}z}\right)}{d-c\left(\frac{\bar{a}+\bar{b}\bar{z}}{b-a\bar{z}}\right)}\right)^s \\
&= \left(\frac{\bar{b}\bar{d}-a\bar{c}-(\bar{a}\bar{d}+b\bar{c})z}{bd-\bar{a}c-(ad+\bar{b}c)\bar{z}}\right)^s, \\
&= \mu_s(UV;z).
\end{aligned} \tag{3.62}
$$

For local sections of the form $f(z) = q^{-s}p(z)$, where $p(z)$ is a polynomial of degree $\leq 2s = n-1$, which are the zero modes on the 2-sphere for $n > 0$ (3.46), we note

$$
\begin{aligned}
(\rho_s(U)f)(z,\bar{z}) &= \left(\frac{\bar{b}-\bar{a}z}{b-a\bar{z}}\right)^s \frac{q^{-s}}{(\bar{b}-\bar{a}z)^{-s}(b-a\bar{z})^{-s}} p\left(\frac{a+bz}{\bar{b}-\bar{a}z}\right), \\
&= \frac{1}{q^s}(\bar{b}-\bar{a}z)^{2s} p\left(\frac{a+bz}{\bar{b}-\bar{a}z}\right),
\end{aligned} \tag{3.63}
$$

where we have used (3.55). Since $p(z)$ has degree $\leq 2s$, this is again a product of q^{-s} with a polynomial of degree $\leq 2s$.

We conclude that the local sections of the form f_1^N in (3.46) form the irreducible representation of $SU(2)$ of dimension $n = 2s + 1$ and spin $j = s$. A similar argument shows that, for $n < 0$, the local sections f_2^N in (3.48) form an irreducible representation of dimension $-n = -2\tilde{s} + 1$ and spin $j = -\tilde{s}$.

3.1.5 Zero-modes on $\mathbb{R}^3$

In this section we show that the zero-modes of the Dirac operator $\not{D}_{S^2,n}$ give rise to zero-modes of a certain massive Dirac operator on Euclidean 3-space. This will provide valuable intuition for analysing the zero-modes on the TN manifold in the next section.

The standard Dirac operator (2.1) in $\mathbb{R}^3$ associated to the flat metric in Cartesian coordinates $ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$, was discussed earlier in Sect. 2.1.3. However, the cartesian form (2.29) is not convenient in the current context for two reasons. The action of rotation of spinors is more complicated in the cartesian frame since it is not rotationally invariant. Furthermore, the monopole gauge potential takes its simpler form in coordinates adapted to the foliation of $\mathbb{R}^3$ into spheres.

Working again on the north pole, we use the metric obtained by adding the radial term dr^2 to the metric on the 2-sphere (3.5),

$$ds^2 = dr^2 + \frac{4r^2}{q^2} dz d\bar{z}, \tag{3.64}$$

which admits the triad

$$e^1 = \frac{2r}{q} dy^1, \quad e^2 = \frac{2r}{q} dy^2, \quad e^3 = dr. \tag{3.65}$$

Then the dual vector fields are

$$E_1^1 = \frac{q}{2r}, \quad E_2^2 = \frac{q}{2r}, \quad E_3^3 = 1. \tag{3.66}$$

The non-vanishing components of the spin connection are

$$\omega_{12} = \frac{2}{q}(y_1 dy_2 - y_2 dy_1), \quad \omega_{23} = \frac{2}{q} dy_2, \quad \omega_{13} = \frac{2}{q} dy_1. \tag{3.67}$$

Working again with the representation $\gamma^j = i\tau^j$ satisfying the relations (2.45) we find

$$\Gamma^{(3)} = \frac{i}{2}(\omega_{12}\tau_3 + \omega_{23}\tau_1 + \omega_{31}\tau_2) = \frac{i}{q}((y_1 dy_2 - y_2 dy_1)\tau_3 + dy_2\tau_1 - dy_1\tau_2). \quad (3.68)$$

Using these we find that the twisted Dirac operator (2.17) coupled to the monopole (2.125) is

$$\not{D}_{\mathbb{R}^3,n} = i\begin{pmatrix} \partial_r + \frac{1}{r} & 0 \\ 0 & -\partial_r - \frac{1}{r} \end{pmatrix} + \frac{1}{r}\not{D}_{S^2,n}, \quad (3.69)$$

where $\not{D}_{S^2,n}$ is defined in (3.16). This decomposition is analogous to (3.4). We will discuss the zero modes of $\not{D}_{\mathbb{R}^3,n}$ in the context of a deformed version of this operator, where the deformation parameter is an inverse length or mass (in units where $\hbar = c = 1$). The operator we consider may be thought of as a singular limit of the Dirac operator coupled to a smooth non-abelian BPS monopole [23]. Callias proved an index theorem for smooth non-abelian BPS monopoles in [24] and considered singular limit where the Higgs field is taken to be constant in [25]. This is the limit we consider here. A different singular limit, first considered in [26], requires the Higgs field to satisfy the abelian Bogomol'nyi equation, see also [27] for a recent discussion of the associated Dirac equation and plots of its zero-modes.

We obtain our operator via dimensional reduction of a Dirac operator in $\mathbb{R}^4$ coupled to a Dirac monopole in $\mathbb{R}^3$ and a constant connection $\frac{i}{\Lambda}dx^4$, where Λ is a non-negative length scale and x^4 a coordinate for the auxiliary fourth dimension. Working again with the coordinates r, z used in (3.64), the metric on $\mathbb{R}^4$ is

$$ds^2 = dr^2 + \frac{4r^2}{q^2}dz d\bar{z} + (dx^4)^2. \quad (3.70)$$

With the Euclidean Dirac matrices

$$\gamma_i = \begin{pmatrix} 0 & \tau_j \\ -\tau_j & 0 \end{pmatrix}, \; j = 1,2,3 \quad \gamma_4 = \begin{pmatrix} 0 & -i\tau_0 \\ -i\tau_0 & 0 \end{pmatrix}, \quad (3.71)$$

we have the commutators

$$[\gamma_4, \gamma_i] = 2i \begin{pmatrix} \tau_i & 0 \\ 0 & -\tau_i \end{pmatrix} \quad \text{and} \quad [\gamma_i, \gamma_j] = -2i\epsilon_{ijk} \begin{pmatrix} \tau_k & 0 \\ 0 & \tau_k \end{pmatrix}. \tag{3.72}$$

Noting that the non-vanishing connection 1-forms are as in (3.67), the spin connection is a 4×4 matrix which can be written in terms of the spin connection $\Gamma^{(3)}$ as

$$\Gamma^{(4)} = \begin{pmatrix} \Gamma^{(3)} & 0 \\ 0 & \Gamma^{(3)} \end{pmatrix}. \tag{3.73}$$

With a $U(1)$ gauge potential which combines the Dirac monopole (2.125) with a constant component in the x_4-direction

$$A = \frac{n}{2q}(zd\bar{z} - \bar{z}dz) + \frac{i}{\Lambda}dx^4, \tag{3.74}$$

the twisted Dirac operator has the general form (2.17). For spinors which do not depend on the auxiliary coordinate x^4 we have $\gamma^j E_j^4(\frac{\partial}{\partial x^4} + A_4 + \Gamma_4^{(4)}) = \frac{i}{\Lambda}\gamma^4$, and the Dirac operator simplifies to

$$\not{D}_{\Lambda,n} = \begin{pmatrix} 0 & -i\not{D}_{\mathbb{R}^3,n} + \frac{1}{\Lambda}\tau_0 \\ i\not{D}_{\mathbb{R}^3,n} + \frac{1}{\Lambda}\tau_0 & 0 \end{pmatrix}. \tag{3.75}$$

From the zero-modes (3.50) of $\not{D}_{S^2,n}$ we can easily obtain the following zero-modes of $\not{D}_{\Lambda,n}$ on the open set $\mathbb{R}^+ \times U_N$:

$$\Psi^N = \frac{e^{-\frac{r}{\Lambda}}}{r} \begin{pmatrix} 0 \\ 0 \\ q^{\frac{1}{2}(1-n)} \sum_{k=0}^{n-1} a_k z^k \\ 0 \end{pmatrix} \text{ if } n \geq 1,$$

$$\Psi^N = \frac{e^{-\frac{r}{\Lambda}}}{r} \begin{pmatrix} 0 \\ q^{\frac{1}{2}(1+n)} \sum_{k=0}^{n-1} \alpha_k \bar{z}^k \\ 0 \\ 0 \end{pmatrix} \text{ if } n \leq -1. \tag{3.76}$$

These solutions are singular at $r = 0$ but square integrable on $\mathbb{R}^3$. When we take the limit $\Lambda = \infty$ we lose the square-integrability. Similarly, allowing for spinors on the 2-sphere which are not zero-modes of $\not{D}_{S^2,n}$ generates solutions which diverge at $r = 0$ faster than $1/r$. Such solutions are also not square-integrable.

We have exhibited an $|n|$-dimensional space of normalisable zero-modes of the deformed or 'massive' Dirac operator (3.75). In the context of this work we are interested in these zero-modes because they provide valuable intuition for understanding the normalisable zero-modes of the twisted Dirac operator on the TN manifold in the next chapter. We do not claim to have proved that all normalisable zero modes are of the form (3.76) although we expect this to be the case. A rigorous discussion would need to address issues of self-adjointness, see [25] for the case of $n = 1$ and [9] for a recent and general treatment of zero-modes of magnetic Dirac operators on $\mathbb{R}^3$.

3.2 Twisted Dirac operators on the Taub-NUT manifold

3.2.1 Dirac operators on self-dual 4-manifolds with rotational symmetry

Although we are primarily interested in the TN manifold in this paper, we initially work in a more general framework and give the form of the Dirac operator for 4-manifolds with isometry group $SU(2)$ or $SO(3)$, acting with generically 3-dimensional orbits, and a self-dual Riemann tensor. A partial list of examples of such 'gravitational instantons' can be found in [17]. In particular, we have in mind the Atiyah-Hitchin manifold which was considered in [1] alongside the TN manifold as a candidate for a geometric model of matter. The metrics can be parametrised in terms of suitable $SU(2)$ or $SO(3)$ orbit parameters (e.g. our Euler angles or complex coordinates) and a transverse, radial coordinate r. The metric can be written in terms of the left-invariant 1-forms σ_j, $j = 1, 2, 3$, and radial functions f, a, b, c:

$$ds^2 = f(r)^2 dr^2 + a(r)^2 \sigma_1^2 + b(r)^2 \sigma_2^2 + c(r)^2 \sigma_3^2. \tag{3.77}$$

The function f may be chosen freely, different choices corresponding to different definitions of the radial coordinate. We introduce the tetrad

$$e^1 = a\sigma_1, \quad e^2 = b\sigma_2, \quad e^3 = c\sigma_3, \quad e^4 = -f dr. \tag{3.78}$$

We use the orientation discussed in [1]. Since the left-invariant 1-forms σ_i, $i = 1, 2, 3$, have the opposite sign of the left-invariant 1-forms used in [1] the resulting volume element is

$$dV = e^1 \wedge e^2 \wedge e^3 \wedge e^4 = fabc\, dr \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3 = -\frac{ab^2c}{r} \sin\beta dr \wedge d\beta \wedge d\alpha \wedge d\gamma. \tag{3.79}$$

The self duality of the Riemann tensor (2.13) with respect to the orientation implies

$$\frac{2bc}{f}\frac{da}{dr} = (b-c)^2 - a^2, \;\; + \text{cycl.}, \tag{3.80}$$

where '+ cycl.' means we add the two further equations obtained by cyclic permutation of a, b, c. Solving (2.8) for the spin connection, we find

$$\begin{aligned} &\omega_{14} = (1-A)\sigma_1, \quad &&\omega_{24} = (1-B)\sigma_2, \quad &&\omega_{34} = (1-C)\sigma_3, \\ &\omega_{23} = -A\sigma_1, &&\omega_{31} = -B\sigma_2, &&\omega_{12} = -C\sigma_3, \end{aligned} \tag{3.81}$$

where

$$A = \frac{b^2+c^2-a^2}{2bc}, \quad B = \frac{a^2+c^2-b^2}{2ac}, \quad C = \frac{a^2+b^2-c^2}{2ab}. \tag{3.82}$$

The vector fields dual to the tetrad (3.78) are

$$E_1 = \frac{1}{a}X_1, \quad E_2 = \frac{1}{b}X_2, \quad E_3 = \frac{1}{c}X_3, \quad E_4 = -\frac{1}{f}\frac{\partial}{\partial r}, \tag{3.83}$$

where X_1, X_2 and X_3 are the left-invariant vector fields on $SU(2)$ (2.72). For our purposes, the advantage of working with the frames (3.78) and (3.83) is that they are rotationally invariant. This results in a choice of gauge for the Dirac operator and the bundle of spinors where the $SU(2)$ action is particularly simple. Note that many treatments of the Dirac operator on the TN manifold (e.g., in [28]) use a different gauge.

For some calculations it is convenient to use a proper radial distance coordinate R defined via

$$dR = f dr, \tag{3.84}$$

and we frequently do this in the remainder of this section. We are interested in the general form of the Dirac operator on metrics like (3.77) and coupled to an spherically symmetric, abelian ($U(1)$ or $\mathbb{R}$) connection with self-dual curvature. Locally, the gauge potential for such a connection can be written in terms of the left-invariant 1-forms as

$$\mathcal{A} = A_1\sigma_1 + A_2\sigma_2 + A_3\sigma_3, \tag{3.85}$$

where A_1, A_2 and A_3 are functions of R only. The curvature is

$$\begin{aligned}\mathcal{F} = d\mathcal{A} =& \frac{1}{a}\frac{dA_1}{dR}e^1 \wedge e^4 - \frac{A_1}{bc}e^2 \wedge e^3 \\ &+ \frac{1}{b}\frac{dA_2}{dR}e^2 \wedge e^4 - \frac{A_2}{ac}e^3 \wedge e^1 + \frac{1}{c}\frac{dA_3}{dR}e^3 \wedge e^4 - \frac{A_3}{ab}e^1 \wedge e^2,\end{aligned} \tag{3.86}$$

which is self-dual if

$$\frac{dA_1}{dR} = -\frac{a}{bc}A_1, \quad \frac{dA_2}{dR} = -\frac{b}{ac}A_2, \quad \frac{dA_3}{dR} = -\frac{c}{ab}A_3. \tag{3.87}$$

In the following we write $D_j = X_j + A_j, j = 1, 2, 3,$ for the associated covariant derivatives.

Working again with the γ-matrices (3.71) which satisfy the relations (3.72), the Dirac operator (2.17) associated to the metric (3.77) and the connection (3.85) takes the form

$$D\!\!\!/_{\mathcal{A}} = \begin{pmatrix} 0 & T^\dagger_{\mathcal{A}} \\ T_{\mathcal{A}} & 0 \end{pmatrix} \tag{3.88}$$

where

$$\begin{aligned}T^\dagger_{\mathcal{A}} &= \frac{i}{f}\frac{\partial}{\partial r} - \frac{i}{2}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + \frac{1}{a}\tau_1 D_1 + \frac{1}{b}\tau_2 D_2 + \frac{1}{c}\tau_3 D_3, \\ T_{\mathcal{A}} &= \frac{i}{f}\frac{\partial}{\partial r} + i\left(\frac{A}{a} + \frac{B}{b} + \frac{C}{c}\right) - \frac{i}{2}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - \frac{1}{a}\tau_1 D_1 - \frac{1}{b}\tau_2 D_2 - \frac{1}{c}\tau_3 D_3.\end{aligned} \tag{3.89}$$

As a result of the rotational (left-) invariance of the metric (3.77) and the connection (3.85), the Dirac operator commutes with the vector fields Z_1, Z_2 and Z_3 (2.85) generating the left-action of $SU(2)$ or $SO(3)$ on the manifold. This is easily checked explicitly, since the left-generators commute with the right generators X_1, X_2 and X_3 and any function of the radial coordinate r, see Sect. 2.2.1 for further details. The operators $iZ_j, j = 1, 2, 3,$ play the role of the total angular momentum operators, combining combining both orbital and spin contributions. In our rotational symmetric gauge, the total angular momentum operators act on the argument of the spinors and do not mix their components.

To check that T and $T^\dagger$ are actually each others adjoints with respect to the L^2 inner product based on the volume element (3.79) we note that, as a consequence of the self-duality equations (3.80),

$$\frac{1}{abcf}\frac{\partial}{\partial r}abc = \frac{A-1}{a} + \frac{B-1}{b} + \frac{C-1}{c} + \frac{1}{f}\frac{\partial}{\partial r}. \tag{3.90}$$

To end this section we show that, for non compact self-dual 4-manifolds, $T^\dagger_{\mathcal{A}}$ has a trivial kernel. This is a special case of a vanishing theorem for Dirac operators on non-compact self-dual manifolds coupled to line boundless with self-dual connections proved in [29]. However, the following short proof for the spherically symmetric case contains some illuminating details. In particular, we see an interesting relation to the Dirac operator on the squashed 3-sphere.

The Dirac operator on the 3-sphere with metric

$$ds^2 = a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2 \tag{3.91}$$

at a fixed value of r (or, equivalently, for real constants a, b and c) and coupled to the connection (3.85) at fixed value of r is

$$\not{D}_{S^3,\mathcal{A}} = \frac{i}{a}\tau_1 D_1 + \frac{i}{b}\tau_2 D_2 + \frac{i}{c}\tau_3 D_3 + \frac{1}{2}\left(\frac{A}{a} + \frac{B}{b} + \frac{C}{c}\right). \tag{3.92}$$

Therefore we can write

$$T_{\mathcal{A}}^{\dagger} = \frac{i}{f}\frac{\partial}{\partial r} - i\not{D}_{S^3,\mathcal{A}} + \frac{i}{2}\left(\frac{A-1}{a} + \frac{B-1}{b} + \frac{C-1}{c}\right),$$
$$T_{\mathcal{A}} = \frac{i}{f}\frac{\partial}{\partial r} + i\not{D}_{S^3,\mathcal{A}} + \frac{i}{2}\left(\frac{A-1}{a} + \frac{B-1}{b} + \frac{C-1}{c}\right). \tag{3.93}$$

We can simplify these expressions by introducing the differentiable function $\nu = \sqrt{|abc|}$, nothing that, for Riemannian metrics, the functions a, b and c solving (3.80) cannot pass through zero and therefore do not change sign. Then, using (3.90), we obtain the symmetric formulae

$$T_{\mathcal{A}} = \frac{i}{\nu}\frac{\partial \nu}{\partial R} + i\not{D}_{S^3,\mathcal{A}}, \quad T_{\mathcal{A}}^{\dagger} = \frac{i}{\nu}\frac{\partial \nu}{\partial R} - i\not{D}_{S^3,\mathcal{A}}, \tag{3.94}$$

and therefore

$$T_{\mathcal{A}}T_{\mathcal{A}}^{\dagger} = -\left(\frac{1}{\nu}\frac{\partial \nu}{\partial R}\right)^2 + \not{D}^2_{S^3,\mathcal{A}} + \frac{\partial \not{D}_{S^3,\mathcal{A}}}{\partial R}. \tag{3.95}$$

Using the self-duality equations (3.80) and (3.87) as well as the commutation relations $[X_i, X_j] = \epsilon_{ijk}X_k$, one finds after a lengthy computation

$$T_{\mathcal{A}}T_{\mathcal{A}}^{\dagger} = -\left(\frac{1}{\nu}\frac{\partial \nu}{\partial R}\right)^2 - \frac{D_1^2}{a^2} - \frac{D_2^2}{b^2} - \frac{D_3^2}{c^2} + \frac{i}{a^2}\tau_1 D_1 + \frac{i}{b^2}\tau_2 D_2 + \frac{i}{c^2}\tau_3 D_3$$
$$+ \left(\frac{a^2+b^2+c^2}{4abc}\right)^2 + \frac{d}{dR}\left(\frac{a^2+b^2+c^2}{4abc}\right). \tag{3.96}$$

Now we observe that

$$\frac{1}{abc}\partial_R abc \partial_R = \left(\frac{1}{\nu}\frac{\partial \nu}{\partial R}\right)^2 - \frac{1}{\nu}\frac{d^2\nu}{dR^2}, \tag{3.97}$$

and complete the square to obtain

$$T_{\mathcal{A}}T_{\mathcal{A}}^{\dagger} = -\frac{1}{abc}\partial_R abc \partial_R - \frac{1}{a^2}\left(D_1 - \frac{i}{2}\tau_1\right)^2 - \frac{1}{b^2}\left(D_2 - \frac{i}{2}\tau_2\right)^2 - \frac{1}{c^2}\left(D_3 - \frac{i}{2}\tau_3\right)^2 + W, \tag{3.98}$$

in which

$$W = -\frac{1}{\nu}\frac{d^2\nu}{dR^2} - \frac{1}{4a^2} - \frac{1}{4b^2} - \frac{1}{4c^2} + \left(\frac{a^2+b^2+c^2}{4abc}\right)^2 + \frac{d}{dR}\left(\frac{a^2+b^2+c^2}{4abc}\right). \tag{3.99}$$

However, this function vanishes identically as a consequence of the self-duality equations (3.80).

Taking the expectation value of the identity (3.98) and integrating by parts, one deduces that any zero-mode of $T_{\mathcal{A}}^{\dagger}$ would have to be covariantly constant. On a non-compact manifold this is impossible for a normalisable spinor. Therefore $T_{\mathcal{A}}^{\dagger}$ cannot have any zero-modes.

3.2.2 Dirac operators on Taub-NUT coupled to self-dual $\mathbb{R}$-gauge fields

We now insert the solution of the self-duality equations (3.80) which gives rise to the TN metric:

$$a = b = r\sqrt{V}, \quad c = \frac{L}{\sqrt{V}}, \quad f = -\frac{b}{r}, \quad V = \epsilon + \frac{L}{r}. \tag{3.100}$$

Here ϵ and L are parameters which are required to be positive for a smooth metric. In this work we set $\epsilon = 1$ so that

$$V = 1 + \frac{L}{r}. \tag{3.101}$$

Substituting into (3.89), we have

$$\begin{aligned} T_{\mathcal{A}}^{\dagger} &= \frac{i}{\sqrt{V}}\left(-\partial_r - \frac{1}{r} - \frac{V}{L}\left(i\tau_3 X_3 + \frac{1}{2}\right) + \frac{1}{r}(-i\tau_1 X_1 - i\tau_2 X_2)\right), \\ T_{\mathcal{A}} &= \frac{i}{\sqrt{V}}\left(-\partial_r - \frac{1}{r} + \frac{V}{L}\left(i\tau_3 X_3 + \frac{1}{2}\right) + \frac{L}{2r^2 V} + \frac{1}{r}\left(i\tau_1 X_1 + i\tau_2 X_2\right)\right). \end{aligned} \tag{3.102}$$

The Dirac operator on the TN manifold has been studied extensively in the literature, starting with [30, 31, 32]. It does not have normalisable zero-modes. However, zero-modes appear when the TN Dirac operator is twisted by an abelian connection with a self-dual curvature, i.e., with a special solution of the Maxwell equations. This connection was first noted and coupled to the Dirac operator by Pope in [7]. Its curvature turns out to have a finite L^2-norm, and has played a role as a BPS state in tests of S-duality [33, 34].

One way to understand the origin of this solution in the TN geometry is to note that the self-duality equations (3.80) for the coefficient functions in the TN case ($a = b$) include the equation

$$2\frac{dc}{dr} = -\frac{fc^2}{ab}, \tag{3.103}$$

which, together with (3.87), implies that

$$\mathcal{A} = Kc^2\sigma_3, \tag{3.104}$$

has a self-dual exterior derivative, for any constant K:

$$\mathcal{F} = d\mathcal{A} = K\frac{c^2}{ab}(e^4 \wedge e^3 + e^2 \wedge e^1) = K\left(\frac{c^3}{ar}dr \wedge \sigma_3 + c^2\sigma_2 \wedge \sigma_1\right), \tag{3.105}$$

where we used $f = -b/r$ and $e_4 = -f dr$. Since $\mathcal{F}$ is exact, it is automatically closed. By self-duality it is co-closed and harmonic.

There are many ways to normalise $\mathcal{F}$, and we will normalise it by picking K so that $\mathcal{A}$ can be interpreted as a connection form on S^3 (viewed as the total space of the Hopf bundle) for large r. With $K = i/(2L^2)$, we have

$$\mathcal{A} = i\frac{c^2}{2L^2}\sigma_3 = \frac{i}{2}\frac{r}{r+L}\sigma_3. \tag{3.106}$$

Taking the limit $r \to \infty$ we obtain the form $\frac{i}{2}\sigma_3$, which, in analogy with (2.119), can be interpreted as a connection 1-form on S^3.

The real 2-form

$$\omega := -\frac{i\mathcal{F}}{2\pi} = \frac{1}{4\pi}\left(\frac{r}{r+L}\sigma_2 \wedge \sigma_1 + \frac{L}{(r+L)^2}dr \wedge \sigma_3\right) \tag{3.107}$$

was tentatively interpreted as the electric field in a geometric model of the electron in [1], where the roles of electric and magnetic fields were swapped relative to the discussion here. In that context, the normalisation $\int_{\mathrm{TN}} \omega \wedge \omega = 1$ was related to the electron charge being -1.

Minimally coupling the connection (3.106) to the Dirac operator, and allowing for spinors with charge $p \in \mathbb{R}$, we obtain the operator

$$\not{D}_p = \begin{pmatrix} 0 & T_p^\dagger \\ T_p & 0 \end{pmatrix}, \tag{3.108}$$

where

$$\begin{aligned}
T_p^\dagger &= \frac{i}{f}\frac{\partial}{\partial r} - \frac{i}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) + \frac{1}{a}\tau_1 X_1 + \frac{1}{b}\tau_2 X_2 + \frac{1}{c}\tau_3\left(X_3 + \frac{ipc^2}{2L^2}\right) \\
&= \frac{i}{\sqrt{V}}\left(-\partial_r - \frac{1}{r} - \frac{V}{2L} + \tau_3\left(\frac{p}{2L} - \frac{iV}{L}X_3\right) - \frac{i}{r}(\tau_1 X_1 + \tau_2 X_2)\right), \\
T_p &= \frac{i}{f}\frac{\partial}{\partial r} + i\left(\frac{A}{a}+\frac{B}{b}+\frac{C}{c}\right) - \frac{i}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) - \frac{1}{a}X_1\tau_1 - \frac{1}{b}X_2\tau_2 - \frac{1}{c}\tau_3\left(X_3 + \frac{ipc^2}{2L^2}\right) \\
&= \frac{i}{\sqrt{V}}\left(-\partial_r - \frac{1}{r} + \frac{V}{2L} + \frac{L}{2r^2V} + \tau_3\left(\frac{iV}{L}X_3 - \frac{p}{2L}\right) + \frac{i}{r}(\tau_1 X_1 + \tau_2 X_2)\right).
\end{aligned} \tag{3.109}$$

Like the Dirac operator (3.88), the Dirac operator (3.108) commutes with the generators Z_1, Z_2 and Z_3 of the $SU(2)$ left-action. The equality $a = b$ for the TN metric further implies that (3.108) also commutes with the right-generator

$$\hat{X}_3 = X_3 - \frac{i}{2}\begin{pmatrix} \tau_3 & 0 \\ 0 & \tau_3 \end{pmatrix}. \tag{3.110}$$

This follows form the identity $[X_3 - \frac{i}{2}\tau_3, (X_1\tau_1 + X_2\tau_2)] = 0$. The operator $\hat{X}_3$ is the lift of the generator X_3 of the central $U(1)$ inside the isometry group $U(2)$ to spinors.

3.2.3 Zero-modes and $SU(2)$ representations

In order to write down the zero modes of (3.108) explicitly, we introduce the dimensionless radial coordinate $\rho = r/L$, so that $V = 1 + 1/\rho$. Further using the notation

$X_\pm = X_1 \pm iX_2$ of Sect. 2.2.1 we have

$$T_p^\dagger = \frac{i}{L\sqrt{V}} \begin{pmatrix} -\partial_\rho - \frac{1}{\rho} - \frac{V}{2} - iVX_3 + \frac{p}{2} & -\frac{i}{\rho}X_- \\ -\frac{i}{\rho}X_+ & -\partial_\rho - \frac{1}{\rho} - \frac{V}{2} + iVX_3 - \frac{p}{2} \end{pmatrix},$$

$$T_p = \frac{i}{L\sqrt{V}} \begin{pmatrix} -\partial_\rho - \frac{1}{\rho} + \frac{V}{2} + \frac{1}{2\rho^2 V} + iVX_3 - \frac{p}{2} & \frac{i}{\rho}X_- \\ \frac{i}{\rho}X_+ & -\partial_\rho - \frac{1}{\rho} + \frac{V}{2} + \frac{1}{2\rho^2 V} - iVX_3 + \frac{p}{2} \end{pmatrix}. \tag{3.111}$$

We are now ready to solve

$$\not{D}_p \Psi = 0 \tag{3.112}$$

for a 4-component spinor Ψ and interpret Pope's formula (1.2) for the dimension of the space of solutions. We will exhibit the zero-modes in our complex notation and decompose them under the action of $SU(2)$. It follows from the discussion in Sect. 3.2.1 that the operator $T_p^\dagger$ has no zero modes. We therefore only need to consider the top two components of Ψ.

The operator T_p commutes with the generators Z_1, Z_2 and Z_3 of the $SU(2)$ left-action and the lifted right-generator $\hat{X}_3$ (3.110). We can therefore assume eigen-spinors to be eigenstates of $Z_3, \hat{X}_3$ and the (scalar) Laplace operator on the round 3-sphere Δ_{S^3}, see (2.87) for an expression in terms of both left- and right-generators of the $SU(2)$ action. These three operators mutually commute, and common eigen-functions are discussed in Sect. 2.2.2. With the eigenvalues of Δ_{S^3} being $-j(j+1)$ for $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, the eigenvalues of m of Z_3 and s of X_3 both lie in the range $-j, -j+1, \ldots, j-1, j$. As explained in Sect. 2.2.2, eigenfunctions can be expressed as homogeneous polynomials in $z_1, z_2, \bar{z}_1, \bar{z}_2$, with holomorphic polynomials for the case $s = j$ and anti-holomorphic polynomials for the case $s = -j$.

Returning to the zero-mode equation (3.112), we first consider the case where only the top component of Ψ is a non-zero function, which we assume to have the factorised form $R(\rho)F(z_1, z_2)$. For this to be a zero-mode, the function $F(z_1, z_2)$ has to be annihilated by X_+ and thus holomorphic in z_1, z_2. It follows that $s = j$ in this case. Fixing j and using (2.106), we deduce the general form of the solution as

$$\Psi(r, z_1, z_2) = \begin{pmatrix} R_j(\rho)\sum_{m=-j}^{j} a_m z_1^{j-m} z_2^{j+m} \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.113}$$

Inserting into (3.112) leads to the radial equation

$$\left(\partial_\rho + \left(\frac{1}{2}(p-1) - j\right) + \left(\frac{1}{2} - j\right)\frac{1}{\rho} - \frac{1}{2\rho(\rho+1)}\right) R_j(\rho) = 0, \tag{3.114}$$

which has the general solution

$$R_j(\rho) = c\frac{\rho^j}{\sqrt{\rho+1}} e^{\left(j - \frac{p-1}{2}\right)\rho}, \tag{3.115}$$

for some constant $c \in \mathbb{C}$. This solution is normalisable provided

$$j < \frac{p-1}{2} \Leftrightarrow 2j + 1 < p, \tag{3.116}$$

which can only happen if $p > 1$.

To find solutions for the case $p < 0$, we consider spinors Ψ where only the second component is non-vanishing and of the form $\tilde{R}(\rho)F(z_1, z_2)$. For this to be a zero-mode, F has to be annihilated by X_-, so has to be anti-holomorphic. It follows that $s = -j$ in this case. Fixing j and using (2.107), we deduce the general form of the solution as

$$\Psi(r, z_1, z_2) = \begin{pmatrix} 0 \\ \tilde{R}_j(\rho)\sum_{m=-j}^{j} \tilde{a}_m \bar{z}_1^{j+m} \bar{z}_2^{j-m} \\ 0 \\ 0 \end{pmatrix}. \tag{3.117}$$

Inserting into (3.112) leads to the radial equation

$$\left(\partial_\rho - \left(\frac{1}{2}(p+1) + j\right) + \left(\frac{1}{2} - j\right)\frac{1}{\rho} - \frac{1}{2\rho(\rho+1)}\right) \tilde{R}_j(\rho) = 0. \tag{3.118}$$

This is the equation (3.114) with p replaced by $-p$. The general solution is therefore

$$\tilde{R}_j(\rho) = \tilde{c}\frac{\rho^j}{\sqrt{\rho+1}}e^{\left(j+\frac{p+1}{2}\right)\rho}, \tag{3.119}$$

for some $\tilde{c} \in \mathbb{C}$. This solution is normalisable provided

$$j < -\frac{p+1}{2} \Leftrightarrow 2j+1 < -p, \tag{3.120}$$

which can only happen if $p < -1$.

Concentrating on the case of $p > 1$, we count zero-modes by noting that the space of solutions for fixed j has dimension $2j+1$. Again using our convention that $[p]$ is the largest integer *strictly* smaller than p (so that [3]=2 etc), the total dimension of the space of zero modes is

$$\dim \ker \not{D}_p = 1 + 2 + \ldots [p] = \frac{1}{2}[p]([p]+1), \tag{3.121}$$

in agreement with Pope's formula (1.2). We now interpret this formula in terms of $SU(2)$ representations and Dirac monopoles.

The action of $U \in SU(2)$ on the zero-modes is simply via pull-back of the action of U^{-1} on z_1, z_2. With the parametrisation of $U \in SU(2)$ in terms of complex numbers a, b satisfying $|a|^2 + |b|^2 = 1$ as in (3.51), the action on (3.113) or (3.117) is

$$U : \Psi(r, z_1, z_2) \mapsto \psi(r, \bar{b}z_1 - \bar{a}z_2, az_1 + bz_2). \tag{3.122}$$

As reviewed in Sect. 2.2.2, the holomorphic (or antiholomorphic) homogeneous polynomials in z_1, z_2 of degree $2j$ form the $(2j+1)$-dimensional irreducible representation of $SU(2)$ under this action. This is precisely the action which we encountered when studying the $SU(2)$ transformations of zero-modes of the twisted Dirac operator on the 2-sphere in (3.56). Thus we conclude that the kernel of $\not{D}_p$ is the sum of irreducible $SU(2)$ representation of dimension $\leq [p]$ or, equivalently, the direct sum of the kernels of the Dirac operators $\not{D}_{S^2,n}$ with $n = 1, 2, \ldots, [p]-1, [p]$.

To understand the latter interpretation better, recall that the TN manifold may be thought of as a static Kaluza-Klein monopole of charge one [35, 36]. In this geometrised description of the magnetic monopole, the $U(1)$ gauge symmetry is encoded in the $U(1)$-right action generated by X_3. Functions, spinors or forms transforming non-trivially under this $U(1)$-action are electrically charged. For spinors, the operator

$$\hat{N} = 2i\hat{X}_3, \tag{3.123}$$

where $\hat{X}_3$, defined in (3.110), is the analogue of the 'Chern-number operator' (3.38) introduced in the context of the twisted Dirac operator on the 2-sphere. It has integer eigenvalues n which count the product of the magnetic and electric charge. The eigenvalue is $n = 2j + 1$ for the solution (3.113) in the case $p > 1$ and is $n = -(2j + 1)$ for the solution (3.117) in the case $p < 1$. As for the Dirac operator $\not{D}_{S^2,n}$, the absolute value of this integer gives the number of zero modes for a fixed n. Summing over all allowed values of j (and hence n) gives all zero modes.

Reverting to the radial coordinate $r = \rho L$, we observe that the radial function in (3.115) and (3.119) plays off exponential growth with coefficient $(2j + 1)/(2L)$ against exponential decay with coefficient $|p|/(2L)$. The exponential growth comes from the geometry of the TN space while the decay comes entirely from the auxiliary $\mathbb{R}$-gauge field. The effective length scale $2L/(|p| - 2j - 1)$ plays a role analogous to that of Λ in the solutions (3.76) of the massive Dirac equation on $\mathbb{R}^3$, but it only has the correct sign if $|p| > 2j + 1$.

We would like to point out that the zero-modes define interesting geometrical shapes in 3-dimensional Euclidean space even though they are defined on the 4-dimensional TN manifold. The reason is that their dependence on the $U(1)$ fibre of TN (viewed as a circle-bundle over $\mathbb{R}^3 \setminus \{0\}$) is a pure phase. Thus, their square - which would give a probability distribution in a hypothetical quantum mechanical interpretation of the zero-modes - only depends on the position in $\mathbb{R}^3$, given by

$$(x_1, x_2, x_3) = (r \sin\beta \cos\alpha, r \sin\beta \sin\alpha, r \cos\beta), \tag{3.124}$$

see also the discussion of the Hopf fibration of section 2.2. Focusing on $p > 1$ and picking a term of fixed m in the zero-mode (3.113), we obtain the axially symmetric distribution

$$|\Psi|^2(x_1, x_2, x_3) \propto \frac{e^{(2j+1-p)\frac{r}{L}}}{r+L}(r-x_3)^{j+m}(r+x_3)^{j-m}. \tag{3.125}$$

For $-j < m < j$, it vanishes along the entire x_3-axis. For $j = m$, it is zero only for $x_3 \geq 0$ while for $j = -m$ it vanishes for $x_3 \leq 0$. We show contour plots of typical zero-modes in Fig. 3.1.

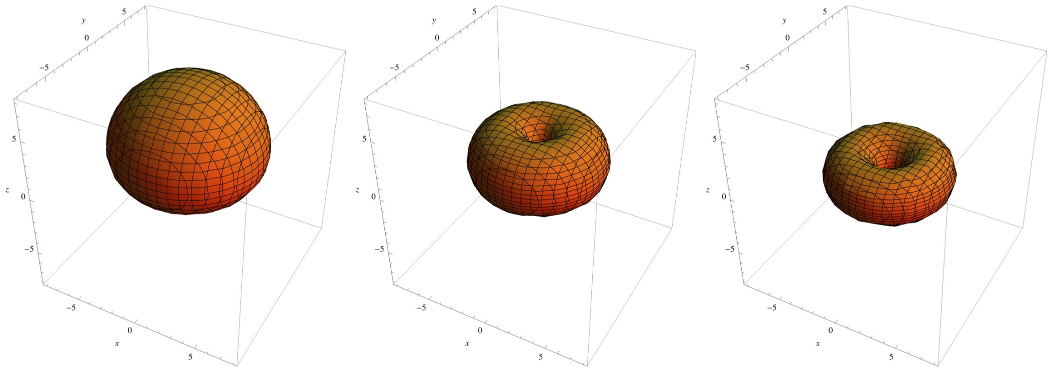

Figure 3.1: Density contours of the squared zero-mode (3.125) for $j = 4$ and $p = 12$ and, from left to right, $m = -4, m = -2, m = 0$

Spin $\frac{1}{2}$

To end our discussion we notice that among all the zero-modes, the spin 1/2 states have a special property. By picking $p = 2$, the spin 1/2 doublet has the functional dependance

$$\sqrt{\frac{r}{r+L}}(a_{-1}z_1 + a_1 z_2), \tag{3.126}$$

which tends to $SU(2)$ doublet states in their standard form $a_{-1}z_1 + a_1 z_2$ as $r \mapsto \infty$. Uniquely among the zero-modes, spin-$\frac{1}{2}$ states can be made to neither to decay to zero nor blow up at spatial infinity by a choice of p. With the same choice of $p = 2$, the spin-0 state is exponentially localised at the origin

$$\frac{e^{-\frac{r}{2L}}}{\sqrt{r+L}}. \tag{3.127}$$

The choice $p = 2$ therefore gives a totally delocalise spin-$\frac{1}{2}$ doublet and an exponentially localised spin-0 singlet.

Chapter 4

Taub-NUT Bound States

4.1 Dynamical symmetries

Having discussed the zero-modes of the twisted Dirac operator on TN, we will now consider the classical and quantum dynamics on TN. We use the Laplace operator as the model of the quantum dynamics and compute its bound states and their corresponding energies. In TN there is a conserved quantity analogous of the Runge-Lenz vector of the Kepler problem, which plays an important role in our discussion. As in the Hydrogen atom, we can use this vector to derive the spectrum of energies algebraically. The angular momentum and Runge-Lenz vectors can be understood as the conserved quantities of an $SO(4)$ action. In order to understand the conserved quantities of TN, we recall some ideas of dynamical symmetries, and review the Kepler problem in some detail.

4.1.1 Dynamical systems

We begin with a review of concepts from symplectic geometry which we use in this chapter.

Recall that a Hamiltonian system can be characterised by a triplet of objects (M, ω, H), where M is an even-dimensional smooth manifold, ω is a non-degenerate closed 2-form and H is a smooth real-valued function on M, called the Hamiltonian. The manifold M is called symplectic and ω the symplectic form. There is an identification between 1-forms and vector fields in M given by ω and since this is non-degenerated the identification is actually a bijection. So each function gives rise to a vector field X_f by the formula

$$df = \omega(X_f,\ ). \tag{4.1}$$

In particular we can associate a vector field X_H to the Hamiltonian via

$$dH = \omega(X_H,\ ). \tag{4.2}$$

For any manifold N, thought of as the configurational space of a mechanical system, the cotangent bundle T^*N carries a canonical 1-form [37] θ whose exterior derivative defines a symplectic structure $\omega = d\theta$ of the phase space $M = T^*N$.

The standard example [38] is the Hamiltonian system $(T^*\mathbb{R}^3, d\theta, H)$, where θ is the 1-form

$$\theta = p_i dx^i, \quad i = 1, 2, 3, \tag{4.3}$$

in which x_i are the coordinates of $\mathbb{R}^3$ and p_i the corresponding momenta $p_i = \dot{x}_i$. In this example

$$\omega = dp_i \wedge dx^i. \tag{4.4}$$

So writing $X_H = X^l \frac{\partial}{\partial x^l} + P^l \frac{\partial}{\partial p_l}$ it follows from (4.2) and (4.4) that

$$\frac{\partial H}{\partial x^i} dx^i + \frac{\partial H}{\partial p_j} dp^j = \delta_{il} P^l dx^i - \delta_{il} X^l dp_i, \tag{4.5}$$

which shows that in this case

$$X_H = \frac{\partial H}{\partial x^i} \frac{\partial}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x^i}. \tag{4.6}$$

The time evolution is generated by the Hamiltonian and so the infinitesimal change in an observable f under this time evolution is given by

$$\dot{f} = X_H f = df(X_H). \tag{4.7}$$

The Poisson bracket associated to the symplectic structure is defined via

$$\{f, g\} = \omega(X_f, X_g) = df(X_g) = -dg(X_f), \tag{4.8}$$

where we have used (4.1). It follows that the Poisson brackets associated to (4.4) are

$$\{f, g\} = \frac{\partial f}{\partial p^i}\frac{\partial g}{\partial x^i} - \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial p^i}. \tag{4.9}$$

Observe that the time evolution of an observable can now be expressed as

$$\dot{f} = \{H, f\}. \tag{4.10}$$

4.1.2 Noether's theorem

In this section we follow [39] to describe the Hamiltonian version of the Noether's theorem.

A transformation of the phase space $\phi : M \to M$ which leaves the symplectic structure invariant. i.e. $\phi^*\omega = \omega$ is called a symplectomorphism (or canonical transformation). Infinitesimally, a vector field X is called symplectic if the Lie derivative of ω in the direction of X vanishes

$$\mathcal{L}_X\omega = 0. \tag{4.11}$$

Using the identity $\mathcal{L}_X = d\iota_X + \iota_X d$ and the fact that the symplectic form ω is closed, the preceding equation is equivalent to

$$d(\iota_X\omega) = 0. \tag{4.12}$$

Locally this implies the existence of a function on the phase space $L \in C^\infty(M)$ such that

$$\omega(X, \ ) = dL. \tag{4.13}$$

In particular if X is a vector field in the configurational space N we can recast the previous result in terms of the canonical 1-form (4.3),

$$\theta(X) = L. \tag{4.14}$$

If the function L can be defined globally on M, the symplectic vector field X is called Hamiltonian vector field.

Now let G be a Lie group acting on M via symplectomorphisms and $\mathfrak{g}$ its Lie algebra. Thus for each $\xi \in \mathfrak{g}$ the associated vector field in M is symplectic. If X_f is also Hamiltonian for all $\xi \in \mathfrak{g}$, then we can pick a function

$$\phi : \mathfrak{g} \to C^\infty(M), \tag{4.15}$$

such that

$$\omega(X_\xi, \ ) = d\phi(\xi), \tag{4.16}$$

and we call the G-action Hamiltonian. If we can choose a function ϕ such that it preserves the Lie algebra

$$\{\phi(\xi), \phi(\eta)\} = \phi([\xi, \eta]), \tag{4.17}$$

then the G-action is called a Poisson action, and the functions $\phi(\xi) \in M$ are called momenta.

We call a G-action on M a symmetry of the dynamical system (M, ω, H) if it is a Poisson action that leaves the Hamiltonian invariant. Thus for all $\xi \in \mathfrak{g}$ we have

$$0 = X_\xi H = dH(X_\xi) = \omega(X_H, X_\xi) = \{\phi(\xi), H\} = -\dot{\phi}(\xi) = 0, \tag{4.18}$$

and so the momentum functions $\phi(\xi)$ are conserved under the time evolution generated by H. This is the Hamiltonian version of Noether's theorem.

4.1.3 The Kepler problem and the 3-sphere

We now apply the ideas of symmetry and moment maps to the Kepler problem. The main point is to show that the angular momentum and Runge-Lenz vectors can be realised as the conserved quantities of a $SO(4)$ action. This is described for example in [40] where it is used Moser's embedding of the Kepler problem in the 3-sphere [12], which is naturally acted on by $SO(4)$. Our treatment includes more details than are available in the literature and provides a useful preparation for the discussion of the TN analogue.

As shown in [12], one can identify the phase space of the Kepler problem $T^*\mathbb{R}^3$ with the cotangent space of the 3-sphere $T^*(S^3)$ via the stereographic projection from the north pole. In this way the Hamiltonian of the Kepler problem

$$H = \frac{|\vec{p}|^2}{2} - \frac{1}{|\vec{r}|}, \tag{4.19}$$

can be realised as the stereographic projection of a energy function on $T^*(S^3)$ and the Kepler orbits on the energy surface $H = -\frac{1}{2}$ can be related to great circles of the 3-sphere. In particular, the collision orbits are related to geodesics on the sphere going through the north pole.

To see how this works we follow [40], and we review some formulae for the stereographic projection $T^*S^n \to T^*\mathbb{R}^n$ that we will employ in the case $n = 3$.

Let $\vec{y} = (y_0, y_1, \ldots, y_n)$ be a real vector, so that $|\vec{y}|^2 = 1$ represents the unit n-sphere. Then defining the tangent vector $\vec{\eta} = (\eta_0, \eta_1, \ldots, \eta_n) = \dot{\vec{y}}$ we can think of T^*S^n as the manifold given by the pairs $(\vec{y}, \vec{\eta})$ satisfying $\vec{y}\cdot\vec{\eta} = 0$, with the canonical 1-form

$$\theta_{S^n} = \vec{\eta} d\vec{y}, \tag{4.20}$$

whose exterior derivative $\omega = d\vec{\eta} \wedge d\vec{y}$ gives a symplectic form of $T^*(S^n)$. Because S^n is a Riemannian manifold we can define [40] an energy function $\mathfrak{h}(\vec{\eta})$ on $T^*(S^n)$ as

$$\mathfrak{h} = \frac{1}{2}|\vec{\eta}|^2. \tag{4.21}$$

With this we have the Hamiltonian system on $T(S^n)$

$$\dot{\vec{y}} = \frac{\partial \mathfrak{h}}{\partial \vec{\eta}}, \quad \dot{\vec{\eta}} = -\frac{\partial \mathfrak{h}}{\partial \vec{y}}. \tag{4.22}$$

From the second equation we see that the geodesics will be solutions of $\mathfrak{h} =$ constant, and so without loss of generality we can pick trajectories with unit velocity $|\vec{\eta}| = 1$, which are restricted to the surface $\mathfrak{h} = \frac{1}{2}$.

The stereographic projection $S^n \to \mathbb{R}^n$ from the north pole is given by

$$\vec{y} \to \vec{w}, \quad w_k = \frac{y_k}{1 - y_0}, \quad k = 1, \ldots, n, \tag{4.23}$$

and the inverse map is

$$y_0 = \frac{|\vec{w}|^2 - 1}{|\vec{w}|^2 + 1}, \quad y_k = \frac{2w_k}{|\vec{w}|^2 + 1}. \tag{4.24}$$

The transformation that carries the canonical 1-form (4.20) into the canonical 1-form for $T^*\mathbb{R}^n$

$$\theta_{\mathbb{R}^n} = \vec{\xi} d\vec{w}, \tag{4.25}$$

is given by

$$\vec{\xi} = (1 - y_0)\vec{\eta} + \eta_0 \vec{y}. \tag{4.26}$$

The inverse relations are

$$\eta_0 = \vec{\xi} \cdot \vec{w}, \quad \vec{\eta} = \frac{1}{2}(1 + |\vec{w}|^2)\vec{\xi} - (\vec{\xi} \cdot \vec{w})\vec{w}. \tag{4.27}$$

Using these in (4.21), the energy function reads

$$\mathfrak{h} = \frac{(|\vec{w}|^2 + 1)^2 |\vec{\xi}|^2}{8}. \tag{4.28}$$

As in (4.2) we can associate to the function $\mathfrak{h}$ a vector field $X_{\mathfrak{h}}$. If we replace $\mathfrak{h}$ by the function $\mathfrak{g}(\mathfrak{h})$ then we would have $d\mathfrak{g}(\mathfrak{h}) = \mathfrak{g}'(\mathfrak{h})d\mathfrak{h}$ and the new vector field $X_{\mathfrak{g}(\mathfrak{h})}$

would be proportional to $X_{\mathfrak{h}}$

$$X_{\mathfrak{g}(\mathfrak{h})} = \mathfrak{g}'(\mathfrak{h}) X_{\mathfrak{h}}. \tag{4.29}$$

More precisely on each surface $\mathfrak{h} =$ constant, the vector field $X_{\mathfrak{g}(\mathfrak{h})}$ is the same as $X_{\mathfrak{h}}$ up to a constant factor $\mathfrak{g}'(\mathfrak{h})$. In particular if $\mathfrak{g}'(\mathfrak{h}) = 1$, the two vector fields agree. By applying this remark to the function $\mathfrak{g}(\mathfrak{h}) = \sqrt{2\mathfrak{h}} - 1$:

$$\mathfrak{g} \circ \mathfrak{h} : (\vec{w}, \vec{\xi}) \to \frac{(|\vec{w}|^2 + 1)|\vec{\xi}|}{2} - 1 \tag{4.30}$$

and the surface $\mathfrak{h} = \frac{1}{2}$, we are able to obtain a new vector $X_{\mathfrak{g}(\mathfrak{h})}$, that coincides with $X_{\mathfrak{h}}$ on the surface $\mathfrak{g}(1/2) = 0$. Observe that we can rewrite

$$\mathfrak{g}(\mathfrak{h}) = \mathfrak{f}\left(H + \frac{1}{2}\right), \tag{4.31}$$

where $\mathfrak{f} = |\vec{\xi}|$ and

$$H = \frac{|\vec{w}|^2}{2} - \frac{1}{|\vec{\xi}|}, \tag{4.32}$$

and so, on the region where $\vec{\xi} \neq 0$, the condition $\mathfrak{g} = 0$ is equivalent to $H = -\frac{1}{2}$. In the case $n = 3$, one can think of H as the Kepler Hamiltonian (4.19) by identifying $\vec{p}$ with $\vec{w}$ and $\vec{r}$ with $\vec{\xi}$. Thus from Kepler point of view, momentum space is S^3 and position space is the cotangent space T^*S^3. Now we observe that

$$X_{\mathfrak{g}(\mathfrak{h})} = \mathfrak{f} X_H, \tag{4.33}$$

on the surface $H = -\frac{1}{2}$. This can be thought of as replacing the time variable t by the new variable s given by $\frac{dt}{ds} = \mathfrak{f}$, which implies [40] that both Hamiltonians generate the same orbits, but traversed at different speeds.

SO(4) action on the 3-sphere

We now use the above formulae for the stereographic projection $T^*S^3 \to T^*\mathbb{R}^3$ to show that the $SO(4)$ action on T^*S^3 induces an action on $T^*\mathbb{R}^3$ whose moment maps are the angular momentum and Runge-Lenz vectors.

To define an $SO(4)$ action on $S^3 \cong SU(2)$ we recall that $SU(2)$ acts on itself by right and left multiplication

$$h \mapsto he^{\varepsilon t_j}, \quad h \mapsto e^{-\varepsilon t_j} h, \quad j = 1, 2, 3, \tag{4.34}$$

where $h \in SU(2)$ and the $t_j = -\frac{i}{2}\tau_j$ (2.54) are the generators of $\mathfrak{su}(2)$. So considering that $SO(4) \cong SU(2) \times SU(2)$ we may define the following $SO(4)$ actions on $h \in SU(2)$

$$h \mapsto lhl^{-1}, \quad h \mapsto lhl, \quad l = e^{i\frac{\varepsilon}{2}\vec{n}\cdot\vec{\tau}}, \tag{4.35}$$

where $\vec{n}$ is a unitary vector. In this case we are going to use a new parametrisation of h in terms of the unitary vector $\vec{y} = (y_0, y_1, y_2, y_3) \in S^3$ as follows $h = y_0\tau_0 + iy_a\tau_a$. Because we are interested in the infinitesimal version of the above action, we only consider terms up to order ε:

$$\begin{aligned}(\tau_0 - i\frac{\varepsilon}{2}\vec{n}\cdot\vec{\tau})(y_0\tau_0 + i\vec{y}\cdot\vec{\tau})(\tau_0 \mp i\frac{\varepsilon}{2}\vec{n}\cdot\vec{\tau}) \;\; &\simeq \;\; y_0\tau_0 + iy_a\tau_a - i\frac{\varepsilon}{2}y_0(\vec{n}\cdot\vec{\tau}) \mp i\frac{\varepsilon}{2}y_0(\vec{n}\cdot\vec{\tau}) \\ &+ \;\; \frac{\varepsilon}{2}(\vec{n}\cdot\vec{\tau})(\vec{y}\cdot\vec{\tau}) \pm \frac{\varepsilon}{2}(\vec{y}\cdot\vec{\tau})(\vec{n}\cdot\vec{\tau}),\end{aligned} \tag{4.36}$$

in which τ_0 is the 2×2 identity as before.

Angular momentum

We now show that the angular momentum is the conserved quantity of the $SU(2)$ action

$$h \mapsto lhl^{-1}, \tag{4.37}$$

which follows from (4.36) by picking the lower sign:

$$\begin{aligned} y_0\tau_0 + i\vec{y}\cdot\vec{\tau} \;\; &\mapsto \;\; y_0\tau_0 + i\vec{y}\cdot\vec{\tau} - \frac{\varepsilon}{2}[\vec{y}\cdot\vec{\tau}, \hat{n}\cdot\vec{\tau}] \\ &= \;\; y_0\tau_0 + i\vec{y}\cdot\vec{\tau} - \frac{\varepsilon}{2}y_a n_b[\tau_a, \tau_b] \\ &= \;\; y_0\tau_0 + i\vec{y}\cdot\vec{\tau} - i(\varepsilon\epsilon_{abc}y_a n_b)\tau_c, \end{aligned} \tag{4.38}$$

where we have used $[\tau_a, \tau_b] = i\epsilon_{abc}\tau_c$. We can recast this as follows

$$y_0 \mapsto y_0, \quad \vec{y} \mapsto \vec{y} - \varepsilon\vec{y} \times \hat{n}. \tag{4.39}$$

Similarly for the unitary tangent vector $\vec{\eta} = \dot{\vec{y}}$, which defines the $SU(2)$ element $\eta_0\tau_0 + i\eta_a\tau_a$, we have the action

$$\eta_0 \mapsto \eta_0, \quad \vec{\eta} \mapsto \vec{\eta} - \varepsilon\vec{\eta} \times \hat{n}. \tag{4.40}$$

We now use the action on $(\vec{y}, \vec{\eta})$ along with the map $T^*S^3 \to T^*\mathbb{R}^3$ to compute the induced action on $(\vec{w}, \vec{\xi})$. So considering (4.23) and (4.39) we see that the action (4.37) becomes

$$w_l \mapsto w_l - \varepsilon\epsilon_{lmk}w_m n_k. \tag{4.41}$$

In the same way, using (4.26) along with (4.40) we obtain

$$\xi_k \mapsto \xi_k - \varepsilon(1 - y_0)\epsilon_{klm}\eta_l n_m - \varepsilon\eta_0\epsilon_{klm}y_l n_m, \tag{4.42}$$

and then using the inverse relations in (4.24) and (4.27) we find

$$\xi_k \mapsto \xi_k - \varepsilon\epsilon_{klm}\xi_l n_m. \tag{4.43}$$

Notice that the action induced on the vectors $\vec{w}, \vec{\xi}$ is just a rotation and so their norm is invariant under this action. We can see that the associated vector field to this action is

$$X_J = -\epsilon_{imk}w_m n_k \frac{\partial}{\partial w_i} - \epsilon_{imk}\xi_m n_k \frac{\partial}{\partial \xi_i}. \tag{4.44}$$

Inserting this into the symplectic 2-form given by the exterior derivative of the canonical 1-form (4.25)

$$\omega = d\theta_{\mathbb{R}^3} = d\xi_l \wedge dw_l, \tag{4.45}$$

yields

$$\begin{aligned} \omega(X_J, \ ) &= \epsilon_{lmk}w_m x_k d\xi_l - \epsilon_{mlk}\xi_l n_k dw_m, \\ &= \epsilon_{lmk}n_k(w_m d\xi_l + \xi_l dw_m) \\ &= d(J_k n_k), \end{aligned} \tag{4.46}$$

in which

$$J_k = \epsilon_{lmk}\xi_l w_m. \tag{4.47}$$

Since we have identified the pair $(\vec{\xi}, \vec{w})$ with $(\vec{r}, \vec{p})$ we therefore deduce that the moment maps are just the components of the angular momentum $\vec{J} = \vec{r} \times \vec{p}$.

Runge-Lenz vector

The computation of the moment maps of the action

$$h \mapsto lhl, \tag{4.48}$$

works in exactly the same way as in the previous case. Thus picking the upper sign in (4.36) we now obtain

$$\begin{aligned} y_0\tau_0 + iy_a\tau_a &\mapsto y_0\tau_0 + iy_a\tau_a - i\varepsilon y_0(\hat{n}\cdot\vec{\tau}) + \frac{\varepsilon}{2}\{\hat{n}\cdot\vec{\tau}, \vec{y}\cdot\vec{\tau}\} \\ &= y_0\tau_0 + iy_a\tau_a - i\varepsilon y_0(\hat{n}\cdot\vec{\tau}) + \frac{\varepsilon}{2}n_a y_b\{\tau^a, \tau^b\} \\ &= y_0\tau_0 + iy_a\tau_a - i\varepsilon y_0 n_a\tau_a + \varepsilon n_a y_a\tau_0, \end{aligned} \tag{4.49}$$

where we have used $\{\tau_a, \tau_b\} = 2\delta_{ab}\tau_0$. Comparing both sides of this relation we find

$$y_0 \mapsto y_0 + \varepsilon\vec{y}\cdot\hat{n}, \quad y_a \mapsto y_a - \varepsilon y_0 n_a, \tag{4.50}$$

and hence

$$\frac{y_a}{1-y_0} \mapsto \frac{y_a - \varepsilon y_0 n_a}{1 - y_0 - \varepsilon\vec{y}\cdot\hat{n}}. \tag{4.51}$$

Recall that for $\delta \ll 1$ we have

$$(a+\delta)^{-1} \simeq \frac{1}{a} - \frac{\delta}{a^2}. \tag{4.52}$$

Thus with $a = 1 - y_0, \delta = -\varepsilon\vec{y}\cdot\hat{n}$ we can write

$$\frac{1}{1 - y_0 - \varepsilon\vec{y}\cdot\hat{n}} \simeq \frac{1}{1-y_0} + \frac{\varepsilon\vec{y}\cdot\hat{n}}{(1-y_0)^2}. \tag{4.53}$$

Inserting this into (4.51) and keeping terms up to order $O(\varepsilon)$

$$\frac{y_a}{1-y_0} \mapsto \frac{y_a}{1-y_0} + \frac{\varepsilon y_a y_b n_b}{(1-y_0)^2} - \frac{\varepsilon y_0 n_a}{1-y_0}. \tag{4.54}$$

Or in terms of $\vec{w}$ (4.23)

$$\vec{w} \mapsto \vec{w} + \varepsilon(\vec{w}\cdot\hat{n})\vec{w} - \frac{\varepsilon}{2}(|\vec{w}|^2 - 1)\hat{n}, \tag{4.55}$$

where we have used the expression for y_0 in (4.24). We continue in the same way to compute the action on $\vec{\xi}$, which follows from (4.26) and the analogous action on the tangent vector $\eta_0\tau_0 + i\eta_a\tau_a$

$$\eta_0 \mapsto \eta_0 + \varepsilon\vec{\eta}\cdot\hat{n}, \quad \eta_a \mapsto \eta_a - \varepsilon\eta_0 n_a. \tag{4.56}$$

Using the above and (4.26) we obtain

$$\xi_a \mapsto \xi_a - \varepsilon\eta_0 n_a - \varepsilon(\vec{y}\cdot\hat{n})\eta_a + \varepsilon(\vec{\eta}\cdot\hat{n})y_a \tag{4.57}$$

which can be recast with the help of (4.24) and (4.27) as follows

$$\xi_a \mapsto \xi_a - \varepsilon(\vec{\xi}\cdot\vec{w})n_a - \varepsilon(\vec{w}\cdot\vec{n})\xi_a + \varepsilon(\vec{\xi}\cdot\hat{n})w_a. \tag{4.58}$$

From the above, we see that the associated vector field to this action is

$$X_M = -\left[-w_i n_i w_k + \frac{1}{2}(w^2-1)n_k\right]\frac{\partial}{\partial w_k} - (\xi_i w_i n_k + w_i n_i \xi_k - \xi_i n_i w_k)\frac{\partial}{\partial \xi_k}. \tag{4.59}$$

The contraction of the symplectic form (4.45) with X_M yields

$$\begin{aligned}\omega(X_M,\ ) &= \left[-w_i n_i w_l + \frac{1}{2}(w^2-1)n_l\right]d\xi_l - (\xi_i w_i n_l + w_i n_i \xi_l - \xi_i n_i w_l)dw_l \\ &= d[-w_i n_i w_j \xi_j + \frac{1}{2}(w^2-1)\xi_i n_i] \\ &= d(\hat{M}_i n_i),\end{aligned} \tag{4.60}$$

where now the moment maps are

$$\hat{M}_i = -\xi_j w_j w_i + \frac{1}{2}(w^2-1)\xi_i. \tag{4.61}$$

Using identification of $(\vec{\xi},\vec{w})$ with $(\vec{r},\vec{p})$ along with the hamiltonian (4.19) we notice that

$$\hat{M}_i = \left[\frac{1}{\sqrt{-2H}}M_i\right]\Big|_{H=-\frac{1}{2}}\,, \tag{4.62}$$

in which M_i are the components of the Runge-Lenz vector of the Kepler problem

$$\vec{M} = \vec{p}\times\vec{J} - \hat{r}. \tag{4.63}$$

Right and Left actions

To end this section we observe that the moment maps (4.47) and (4.61) satisfy the algebra

$$\begin{aligned}\{J_i, J_j\} &= \epsilon_{ijk}J_k,\\ \{J_i, \hat{M}_j\} &= \epsilon_{ijk}\hat{M}_k,\\ \{\hat{M}_i, \hat{M}_j\} &= \epsilon_{ijk}J_k.\end{aligned} \tag{4.64}$$

To see that this is actually the Lie algebra of $SO(4)\cong SU(2)\times SU(2)$ one can use the above relations to check that the quantities

$$R_i = \frac{1}{2}(J_i - \hat{M}_i), \quad L_i = \frac{1}{2}(J_i + \hat{M}_i), \tag{4.65}$$

generate two copies of $SU(2)$,

$$\begin{aligned}\{R_i, R_j\} &= \epsilon_{ijk}R_k,\\ \{L_i, L_j\} &= \epsilon_{ijk}L_k,\\ \{R_i, L_j\} &= 0.\end{aligned} \tag{4.66}$$

The quantities R_i and L_i are the moment maps of the right $h\mapsto hl$ and left actions $h\mapsto lh$ respectively.

We check that both X_J and X_M preserve the hypersurface $H=-\frac{1}{2}$, which is therefore $SO(4)$ invariant.

4.2 A toy model: motion on a surface with magnetic field

We can gain a qualitative understanding of bound states on TN coupled to a Maxwell field by considering a 2-dimensional model, consisting of a 2-dimensional manifold with metric and magnetic field. We will encounter a manifold and metric of the same kind in our study of TN as a geodesic submanifold, and the magnetic field as the restriction of the Maxwell field to the geodesic submanifold. However, here we study the 2-dimensional model in its own right.

Consider a 2-dimensional manifold diffeomorphic to an open disk D with $U(1)$-invariant metric of the form

$$ds^2 = dR^2 + c^2(R)d\gamma^2. \tag{4.67}$$

For consistency with our later discussion of the TN geometry we take the angular coordinate γ in the interval $[0, 4\pi)$, so that $4\pi c$ is the length of a $U(1)$ orbit. The radial coordinate R is the proper radial distance from the origin and has range $[0, \infty)$, and we assume a form of c near $R = 0$ to ensure that the metric is smooth there. We are interested in two kinds of behaviour of the function c.

The first case captures what happens in the regular TN geometry. The function c has the finite range $[0, L)$ for some positive real number L so that the length of the $U(1)$ orbits remain bounded. Moreover we assume that $c(0) = 0$ and that c is strictly monotonic, so that one can picture the metric as being induced on a cigar-shaped surface of revolution in 3-dimensional Euclidean space, as shown in Fig. 4.1. The qualitative behaviour of geodesics on such a surface is well know and follows from Clairaut's relation. Generic geodesics spiral on the cigar. Geodesics spiralling towards the tip will be reflected at some point and spiral out. All geodesics ultimately move arbitrarily far away from the tip and there are no geodesics which remain in a region bounded by a finite value of R.

The second case captures what happens in the singular or 'negative mass' TN.

The function c diverges at $R = 0$, has the range (L, ∞) and is monotonically decreasing. As an embedded surface, this is a funnel, with the opening at $R = 0$ and the tip at $R = \infty$ as shown in Fig. 4.1. Generic geodesics again spiral on this surface, but now there are two kinds of behaviour. Geodesics which travel straight down the funnel or spiral only slowly may escape to $R = \infty$. However, geodesics travelling into the funnel with sufficiently high angular momentum relative to their speed will bounce back and remain inside a region bounded by some finite value of R.

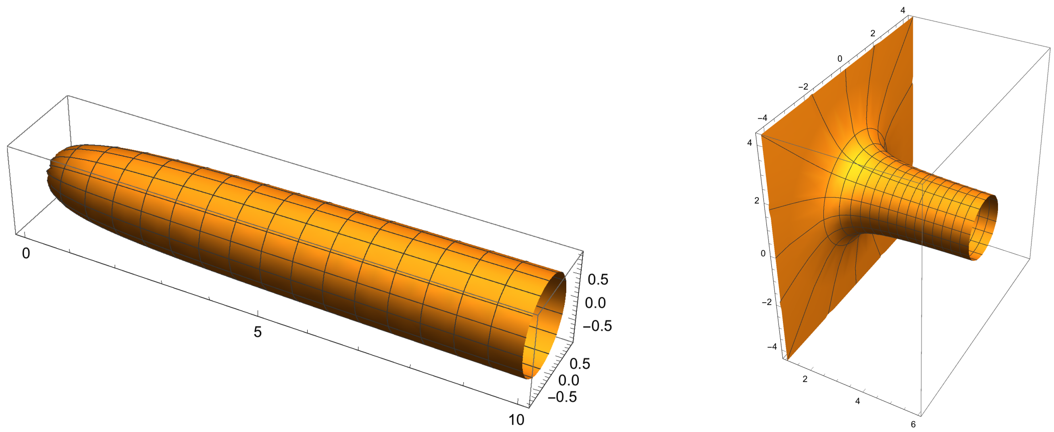

Figure 4.1: The cigar-shaped surface for positive L (left) and the funnel-shaped surface for negative L (right).

We now return to the first case with monotonically increasing $c \in [0, L)$ and consider the inclusion of a magnetic field of a specific type given by the 2-form

$$B = d\left(\frac{pc^2}{2L^2}d\gamma\right) = \frac{p}{L^2}c\,dc \wedge d\gamma, \tag{4.68}$$

for some real constant p which controls the strength of the magnetic field, and is proportional to its flux:

$$\frac{1}{2\pi}\int_D B = p. \tag{4.69}$$

The Lagrangian governing the motion of a particle on the surface with metric (4.67), minimally coupled to the gauge potential for B is, for a suitably chosen mass parameter,

$$\mathcal{L} = \frac{1}{4}\left(\dot{R}^2 + c^2\dot{\gamma}^2\right) - \frac{pc^2}{2L^2}\dot{\gamma}. \tag{4.70}$$

With the momenta conjugate to R and γ

$$p_R = \frac{\partial \mathcal{L}}{\partial \dot{R}} = \frac{1}{2}\dot{R} \qquad q = \frac{\partial \mathcal{L}}{\partial \dot{\gamma}} = \frac{1}{2}c^2\dot{\gamma} - \frac{pc^2}{2L^2}, \tag{4.71}$$

the Hamiltonian is obtained by the Legendre transformation

$$\begin{aligned} H &= \dot{R}p_R + \dot{\gamma}q - \mathcal{L} \\ &= p_R^2 + \left(\frac{q}{c} + \frac{pc}{2L^2}\right)^2, \end{aligned} \tag{4.72}$$

Since q is conserved and p constant, this is effectively the Hamiltonian for 1-dimensional motion on the half-line in the potential

$$W = \left(\frac{q}{c} + \frac{pc}{2L^2}\right)^2. \tag{4.73}$$

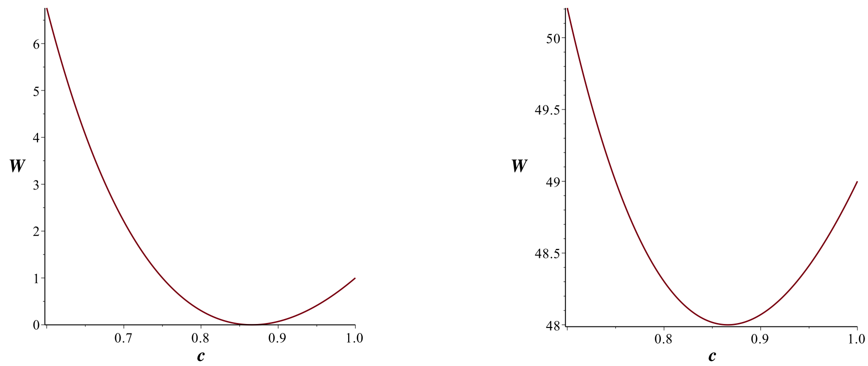

Figure 4.2: Plots of the potential (4.73) for $L = 1$ and $c \in [0, 1)$ for $q = 3$ and $p = -8$ (left) and $q = 3$ and $p = 8$ (right).

We would like to know if there are bounded trajectories in the potential (4.73). As a potential, W should be viewed as a function of R, but with our assumption that $dc/dR > 0$ we can study its minima by looking at W as a function of c. It is easy to check that $W(c)$ has a unique minimum at $c_m > 0$ satisfying

$$\frac{c_m^2}{L^2} = \left|\frac{2q}{p}\right|. \tag{4.74}$$

However, for c_m to be in the range $[0, L)$ we require

$$|q| < \left|\frac{p}{2}\right|, \tag{4.75}$$

and this is a necessary and sufficient condition for W to have a minimum. The value at the minimum is

$$W(c_m) = \begin{cases} 0 & \text{if } pq < 0 \\ \frac{2pq}{L^2} & \text{if } pq \geq 0. \end{cases} \tag{4.76}$$

The qualitative form of the potential is similar in the two cases. We also note that the inequality (4.76) implies the bound

$$L^2 W \geq \begin{cases} 0 & \text{if } pq < 0 \\ 2pq & \text{if } pq \geq 0, \end{cases} \tag{4.77}$$

which will play an important role in our discussion.

We conclude that a magnetic field on a cigar-shaped surface can lead to bounded trajectories even though the geometry of the cigar does not support any bounded geodesics. In the presence of the magnetic field (4.68), all trajectories with angular momentum q of magnitude less than $|p/2|$ remain in a bounded region.

4.3 Twisted Laplace operator in Taub-NUT

4.3.1 The gauged Dirac and Laplace operators

In this section we derived a twisted Laplace operator associated to the Dirac operator (3.108) in the TN space M_{TN} coupled to the abelian gauged (3.106) whose curvature $\mathcal{F}$

$$\mathcal{F} = d\mathcal{A} = \frac{ip}{L^2}\left(cdc \wedge \sigma_3 + \frac{c^2}{2}\sigma_2 \wedge \sigma_1\right), \tag{4.78}$$

is (up to a multiplicative constant) the unique harmonic, normalisable and $U(2)$-invariant 2-form on TN.

Note that the gauge we have chosen preserves the $U(2)$ symmetry of the TN geometry. However, the magnetic field inevitably breaks the discrete symmetry

$$\beta \mapsto \pi - \beta, \quad \alpha \mapsto \pi + \alpha, \quad \gamma \mapsto -\gamma, \tag{4.79}$$

which maps

$$\sigma_1 \mapsto \sigma_1, \quad \sigma_2 \mapsto -\sigma_2, \quad \sigma_3 \mapsto -\sigma_3, \tag{4.80}$$

and therefore preserves the metric (3.77) but neither the gauge field (3.106) nor its curvature. As we will see, this has interesting consequences in the dynamics.

TN without the point $r = 0$ (the NUT) is a circle bundle over $\mathbb{R}^3 \setminus \{0\}$. As reviewed in [5], for each direction (β, α) in the base there is a geodesic submanifold parametrised by (r, γ). Each of these geodesic submanifolds is of the general cigar-shape of our toy model in Sect. 2, and the flux of the magnetic field (4.78) through this submanifold is $2\pi p$. Thinking of TN as a 2-sphere's worth of such cigar-shaped surfaces threaded by magnetic flux will prove very helpful for a qualitative understanding of our results.

The most fundamental operator associated to the metric (3.77) and the connection (3.106) is the Dirac operator on TN minimally coupled to the connection. As shown in [28] for the ungauged case, the spectrum of the Dirac operator and the Laplace operator are closely related. The arguments in [28] are essentially a reflection of an underlying supersymmetry. It is not difficult to adapt them to the gauged case, as we shall now show.

In terms of the variable $r = L\rho$, the components (3.111) of the Dirac operator (3.108) read

$$\begin{aligned} T_p^\dagger &= \frac{i}{\sqrt{V}}\left(-\partial_r - \frac{1}{r} - \frac{V}{2L} + \tau_3\left(\frac{p}{2L} - \frac{iV}{L}X_3\right) - \frac{i}{r}(\tau_1 X_1 + \tau_2 X_2)\right), \\ T_p &= \frac{i}{\sqrt{V}}\left(-\partial_r - \frac{1}{r} + \frac{V}{2L} + \frac{L}{2r^2 V} + \tau_3\left(\frac{iV}{L}X_3 - \frac{p}{2L}\right) + \frac{i}{r}(\tau_1 X_1 + \tau_2 X_2)\right). \end{aligned} \tag{4.81}$$

As shown in [7, 10] and elaborated in Sect. 3.2.1, the kernel of $T_p^\dagger$ is trivial but that of T_p has dimension (3.121). It follows that

$$H_- = T_p T_p^\dagger \tag{4.82}$$

is a strictly positive operator, and that

$$H_+ = T_p^\dagger T_p \tag{4.83}$$

is positive, but not strictly positive. Therefore, we can define the unitary operator

$$U = \frac{1}{\sqrt{H_-}} T_p, \tag{4.84}$$

with inverse $U^{-1} = T_p^\dagger/\sqrt{H_-}$, and use it to relate H_+ and H_- via

$$H_+ = U^{-1} H_- U. \tag{4.85}$$

It follows from this unitary equivalence that, as in the ungauged case [28], H_+ has the same spectrum as H_- apart from the zero eigenvalue of H_-. In other words if Ψ is a 2-component eigenspinor of H_- with eigenvalue E then $U^{-1}\Psi$ is an eigenspinor of H_+ with the same eigenvalue.

Combining these results, we obtain eigenstates of the Dirac operator with non-zero eigenvalues from eigenstates of H_- as follows. Suppose that Ψ is an eigenstate of H_- with eigenvalue $E > 0$. Then

$$\not{D} \begin{pmatrix} U^{-1}\Psi \\ \pm\Psi \end{pmatrix} = \pm\sqrt{E} \begin{pmatrix} U^{-1}\Psi \\ \pm\Psi \end{pmatrix}. \tag{4.86}$$

Inserting the expressions for the TN profile functions (3.100), we find

$$H_- = T_p T_p^\dagger = -\frac{1}{r^2 V}\partial_r(r^2\partial_r) - \frac{1}{r^2 V}\left[(X_1+t_1)^2 + (X_2+t_2)^2\right] - \frac{V}{L^2}\left(X_3 + \frac{ip}{2V} + t_3\right)^2. \tag{4.87}$$

It is convenient to change gauge by observing from (2.70) that

$$X_i h^{-1} = -t_i h^{-1} \tag{4.88}$$

and that therefore, as an operator identity,

$$h(X_i + t_i)h^{-1} = h(X_i h^{-1}) + hh^{-1}X_i + ht_i h^{-1} = X_i. \tag{4.89}$$

Employing this, we obtain

$$hH_{-}h^{-1} = H_p\tau_0, \tag{4.90}$$

where τ_0 is the 2×2 identity matrix and

$$H_p = -\frac{1}{r^2V}\partial_r(r^2\partial_r) - \frac{1}{r^2V}(X_1^2 + X_2^2) - \frac{V}{L^2}\left(X_3 + \frac{ip}{2V}\right)^2 \tag{4.91}$$

is the Laplace operator associated to the metric (3.77) and minimally coupled to the gauge field (3.106). This is the operator whose spectrum we shall study in the remainder of this chapter.

4.4 Dynamical symmetries in classical Taub-NUT dynamics

4.4.1 Canonical procedure

We now turn our attention to the classical dynamics in TN in the gauged case. We discuss the conserved angular momentum and Runge-Lenz vectors, and use them to describe the classical trajectories. Our treatment is an extension of the discussion in [3] and [41] of the (ungauged) motion on TN space.

As reviewed in Sect. 2.2.1 the invariant 1-forms σ_i of S^3 can be parametrised by the Euler angles α, β and γ. In these coordinates and the radial coordinate r the Lagrangian for geodesic motion associated to the metric (3.77) takes the form

$$\mathcal{L} = \frac{1}{4}(f^2\dot{r}^2 + a^2\omega_1^2 + b^2\omega_2^2 + c^2\omega_3^2), \tag{4.92}$$

where ω_i are the components of the body fixed angular velocity,

$$\begin{aligned}
\omega_1 &= \sin\gamma\,\dot{\beta} - \cos\gamma\sin\beta\,\dot{\alpha}, \\
\omega_2 &= \cos\gamma\,\dot{\beta} + \sin\gamma\sin\beta\,\dot{\alpha}, \\
\omega_3 &= \dot{\gamma} + \cos\beta\,\dot{\alpha},
\end{aligned} \tag{4.93}$$

and we have chosen an overall factor of 1/4 for convenience.

Inserting the angular velocities in the Lagrangian and recalling that $a = b = r\sqrt{V}, c = L/\sqrt{V}, f = -b/r$ we obtain

$$\mathcal{L} = \frac{1}{4}\left[V(\dot{r}^2 + r^2\dot{\beta}^2 + r^2\sin^2\beta\dot{\alpha}^2) + L^2V^{-1}(\dot{\gamma} + \cos\beta\dot{\alpha})^2\right]. \tag{4.94}$$

In terms of cartesian coordinates

$$\vec{r} = (x_1, x_2, x_3) = (r\sin\beta\cos\alpha, r\sin\beta\sin\alpha, r\cos\beta), \tag{4.95}$$

the Lagrangian takes the more familiar form

$$\mathcal{L} = \frac{1}{4}(V|\dot{\vec{r}}|^2 + L^2V^{-1}(\dot{\gamma} + \vec{A}\cdot\dot{\vec{r}})^2), \tag{4.96}$$

where $\vec{A}$ is a gauge potential for the Dirac monopole $\vec{A}\cdot d\vec{r} = \cos\beta\, d\alpha$ (see Sect. (2.2.4)), whose components in the coordinates (4.95)

$$A_1 = -\frac{x_3x_2}{r(r^2 - x_3^2)}, \quad A_2 = \frac{x_3x_1}{r(r^2 - x_3^2)}, \quad A_3 = 0, \tag{4.97}$$

satisfy $\nabla \times \vec{A} = \vec{\nabla}\frac{1}{r}$,

$$\partial_l A_m - \partial_m A_l = -\epsilon_{klm}\frac{x_k}{r^3} \quad \text{for} \ \ r \neq 0, \tag{4.98}$$

as well as

$$\vec{A}\cdot d\vec{r} = \cos\beta d\alpha. \tag{4.99}$$

We now minimally couple the motion on TN to the gauge potential (3.106) via the Lagrangian

$$\mathcal{L}_p = \frac{V}{4}|\dot{\vec{r}}|^2 + \frac{L^2}{4V}(\dot{\gamma} + \vec{A}\cdot\dot{\vec{r}})^2 - \frac{p}{2V}(\dot{\gamma} + \vec{A}\cdot\dot{\vec{r}}). \tag{4.100}$$

Clearly, the momentum q conjugate to the cyclic coordinate γ,

$$q = \frac{\partial\mathcal{L}_p}{\partial\dot{\gamma}} = \frac{L^2}{2V}(\dot{\gamma} + \vec{A}\cdot\dot{\vec{r}}) - \frac{p}{2V}, \tag{4.101}$$

is conserved in virtue of the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}_{\mathcal{A}}}{\partial \dot{\gamma}}\right) = \frac{\partial \mathcal{L}_{\mathcal{A}}}{\partial \gamma}. \tag{4.102}$$

The canonical momentum $\vec{\pi}$ conjugate to $\vec{r}$ is

$$\vec{\pi} = \frac{\partial}{\partial \dot{\vec{r}}}\mathcal{L}_p = \frac{1}{2}V\dot{\vec{r}} + \frac{L^2}{2V}(\dot{\gamma} + \vec{A}\cdot\dot{\vec{r}})\vec{A} - \frac{p}{2V}\vec{A} = \vec{p} + q\vec{A}, \tag{4.103}$$

where

$$\vec{p} = \frac{1}{2}V\dot{\vec{r}} \tag{4.104}$$

is called the mechanical momentum [3].

The canonical symplectic structure on the phase space T^*M_{TN},

$$dx_l \wedge d\pi_l + d\gamma \wedge dq, \tag{4.105}$$

is invariant under the $U(1)$ action which maps $\gamma \to \gamma + \delta$. The moment map for this action is the charge q, viewed as map $T^*M_{\text{TN}} \to \mathbb{R}$, and the symplectic quotient by this $U(1)$ action

$$\mathcal{M}_q = T^*M_{\text{TN}}/\!\!/U(1) \tag{4.106}$$

is, by definition, the pre-image of any real constant under the map q divided by the $U(1)$ action. The position vector $\vec{r}$ and the canonical momentum vector $\vec{\pi}$ provide natural coordinates in terms of which the symplectic structure on $\mathcal{M}_q$ takes the form

$$\omega_{TN} = dx_l \wedge d\pi_l = dx_l \wedge dp_l + q dx_l \wedge dA_l. \tag{4.107}$$

Observing that

$$
\begin{aligned}
dx_l \wedge dA_l &= \partial_i A_l dx_l \wedge dx_i \\
&= \frac{1}{2}(\partial_i A_l dx_l \wedge dx_i + \partial_i A_l dx_l \wedge dx_i) \\
&= \frac{1}{2}(\partial_i A_l - \partial_l A_i) dx_l \wedge dx_i \\
&= \frac{1}{2r^3}\epsilon_{iln} x_n dx_i \wedge dx_l, \qquad (4.108)
\end{aligned}
$$

where we have used (4.98) in the last step, then we have

$$\omega_{TN} = dx_l \wedge dp_l + \frac{q}{2r^3}\epsilon_{iln} x_n dx_i \wedge dx_l. \qquad (4.109)$$

The associated Poisson brackets of (4.107) (see Sect. 4.1.1) are

$$\{A, B\} = \frac{\partial A}{\partial x_l}\frac{\partial B}{\partial \pi_l} - \frac{\partial A}{\partial \pi_l}\frac{\partial B}{\partial x_l}, \qquad (4.110)$$

so that the mechanical momentum $\vec{p} = \vec{\pi} - q\vec{A}$ satisfies

$$\{p_i, p_j\} = -q\epsilon_{ijk}\frac{x_k}{r^3}, \quad \{p_i, f(\vec{r})\} = -\partial_i f(\vec{r}), \qquad (4.111)$$

where f is any function of $\vec{r}$.

We now rewrite the Lagrangian in terms of $\vec{p}$ and q,

$$\mathcal{L}_p = \frac{1}{V}|\vec{p}|^2 + \frac{q^2}{L^2}V - \frac{p^2}{4L^2V}, \qquad (4.112)$$

and perform the Legendre transformation to obtain the gauged Hamiltonian,

$$
\begin{aligned}
H_p &= \dot{\vec{r}} \cdot \vec{\pi} + \dot{\gamma} q - \mathcal{L}_p \\
&= \frac{1}{V}|\vec{p}|^2 + \frac{q^2}{L^2}V + \frac{pq}{L^2} + \frac{p^2}{4L^2V} \\
&= H + \Delta H. \qquad (4.113)
\end{aligned}
$$

Here H is the Hamiltonian for $p = 0$ and ΔH the contribution of the gauge potential:

$$H = \frac{1}{V}|\vec{p}|^2 + \frac{q^2}{L^2}V, \quad \Delta H = \frac{pq}{L^2} + \frac{p^2}{4L^2V}. \tag{4.114}$$

Recalling that the profile function c appearing in the TN metric is $c = L/\sqrt{V}$, we note that

$$\begin{aligned} H_p &= \frac{1}{V}|\vec{p}|^2 + \left(\frac{q}{c} + \frac{pc}{2L^2}\right)^2 \\ &\geq \left(\frac{q}{c} + \frac{pc}{2L^2}\right)^2 \\ &\geq \begin{cases} 0 & \text{if } pq < 0 \\ \frac{2pq}{L^2} & \text{if } pq \geq 0. \end{cases} \end{aligned} \tag{4.115}$$

For the last step we observed that the second term in the first line is the potential (4.73) of the toy model of Sect. 4.2, and used the bound (4.77).

There is a conserved angular momentum [3] of H given by

$$\vec{J} = \vec{r} \times \vec{p} + q\hat{r}, \tag{4.116}$$

which, by virtue of (4.111), satisfies the relations

$$\{J_k, p_l\} = \epsilon_{klm} p_m. \tag{4.117}$$

It also follows that

$$\{J_k, J_l\} = \epsilon_{klm} J_m. \tag{4.118}$$

Relation (4.117) can be employed to check that $\vec{J}$ Poisson commutes with the Hamiltonian H. Since ΔH is spherically symmetric, $\vec{J}$ also commutes with $H_p = H + \Delta H$.

In their study of the geodesic motion on the negative mass TN space in [3], Gibbons and Manton showed that there is a conserved vector quantity analogous to the Runge-Lenz vector of the Kepler problem which takes the form

$$\vec{M} = \vec{p} \times \vec{J} - \frac{\hat{r}}{2L}\left(L^2 H - 2q^2\right). \tag{4.119}$$

One checks that it satisfies

$$\{J_k, M_l\} = \epsilon_{klm} M_m, \tag{4.120}$$

and commutes with the TN Hamiltonian H for any value (positive or negative) of L. However, it fails to commute with our gauged Hamiltonian H_p since

$$\{\Delta H, M_k\} = -\frac{p^2}{4LrV^2} p_k + \frac{p^2}{4Lr^3V^3} x_k (\vec{r}\cdot\vec{p}). \tag{4.121}$$

By trial and error we find that the vector-valued function

$$\vec{f} = \frac{p^2 \vec{r}}{8LrV}, \tag{4.122}$$

satisfies $\{H_p, f_k\} = \{\Delta H, M_k\}$. Hence the components of the gauged Runge-Lenz vector

$$\vec{M}^p = \vec{M} - \vec{f} = \vec{p}\times\vec{J} - \frac{\hat{r}}{2L}\left(L^2 H_p - 2q^2 - pq\right), \tag{4.123}$$

commute with H_p. The Poisson brackets between the components of $\vec{J}$ and $\vec{M}^p$ turn out to be

$$\begin{aligned} \{J_i, M_j^p\} &= \epsilon_{ijk} M_k^p, \\ \{M_i^p, M_j^p\} &= \left[\frac{1}{L^2}\left(q + \frac{p}{2}\right)^2 - H_p\right]\epsilon_{ijk} J_k. \end{aligned} \tag{4.124}$$

We will study their Lie-algebraic interpretation in detail in Sects. 4.6 and 4.7.

4.4.2 Classical trajectories

The conserved quantities discussed above can be used to determine the classical trajectories on TN in the gauged situation, i.e., the solutions of the Euler-Lagrange equations of (4.100) or Hamilton equations of (4.113) with Poisson brackets (4.110). Considering first the simpler case where $q = 0$, we deduce from (4.116) and (4.123) that

$$\vec{J}\cdot\hat{r} = 0, \quad \vec{J}\cdot\vec{M}^p = 0, \quad \vec{M}^p\cdot\vec{r} = J^2 - \frac{1}{2}LEr, \tag{4.125}$$

where E denotes the (constant) value of H_p and $J = |\vec{J}|$. The first and second equations show that the movement is in a plane orthogonal to $\vec{J}$ and that $\vec{M}^p$ is in this plane. Using polar coordinates (r, ϕ) to parametrise the plane with $\vec{M}^p$ in the direction determined by $\phi = 0$, we deduce, from the third equation,

$$r = \frac{J^2}{|\vec{M}^p| \cos\phi + \frac{1}{2} L E}. \tag{4.126}$$

This is the equation of a conic section. Finally taking into account the relation

$$|\vec{M}^p| = \sqrt{\left(E - \frac{p^2}{4L^2}\right) J^2 + \frac{1}{4} L^2 E^2}, \tag{4.127}$$

we obtain the following types of orbits: the conic section is an ellipse for $L^2 E < \frac{p^2}{4}$, a parabola for $L^2 E = \frac{p^2}{4}$ and a hyperbola for $L^2 E > \frac{p^2}{4}$.

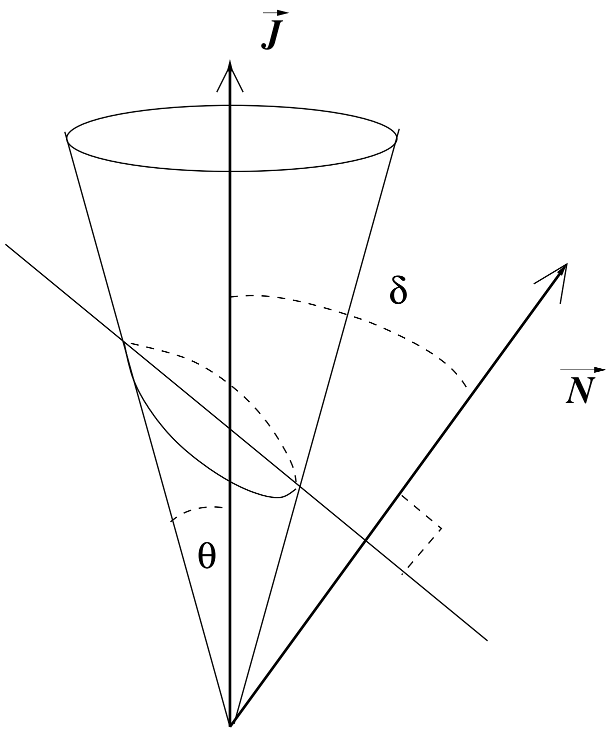

Figure 4.3: The conic sections determined by the conserved vectors $\vec{J}$ and $\vec{N}$.

In the general case $q \neq 0$, the expression (4.116) implies

$$\vec{J} \cdot \hat{r} = q, \tag{4.128}$$

which shows that $\vec{r}$ lies on a cone whose axis of symmetry is along $\vec{J}$ and whose vertex is at the origin. The opening angle $2\theta \in (0, \pi)$ of the cone relative to direction of $\vec{J}$ as shown in Fig. 4.3 is determined by

$$\cos\theta = \frac{|q|}{J}. \tag{4.129}$$

For $q > 0$, the cone is in the 'positive' half-space determined by $\vec{J} \cdot \vec{r} > 0$, while for $q < 0$ it is in the 'negative' half-space determined by $\vec{J} \cdot \vec{r} < 0$.

Furthermore, the equations (4.116) and (4.123) imply

$$\vec{J} \cdot \vec{M}^p = -\frac{q}{2L}\left(L^2 E - 2q^2 - pq\right), \quad \vec{M}^p \cdot \vec{r} = J^2 - q^2 - \frac{r}{2L}\left(L^2 E - 2q^2 - pq\right). \tag{4.130}$$

To interpret them, we define the vector

$$\vec{N} = q\vec{M}^p + \frac{1}{2L}\left(L^2 E - 2q^2 - pq\right)\vec{J}. \tag{4.131}$$

As a linear combination of conserved vectors with conserved coefficients, this vector is also conserved. In terms of this vector, the second equation in (4.130) is equivalent to

$$\vec{N} \cdot \vec{r} = q(J^2 - q^2), \tag{4.132}$$

which shows that the motion is also in a plane perpendicular to the vector $\vec{N}$. With the notation $l = |\vec{r} \times \vec{p}|$ for the magnitude of the orbital angular momentum, we note

$$J^2 = l^2 + q^2, \tag{4.133}$$

so that

$$\vec{N} \cdot \vec{r} = ql^2. \tag{4.134}$$

The classical trajectories in the case $q \neq 0$ are thus intersections of the cone defined by (4.128) and the plane defined by (4.134). From classical geometry we know that these are ellipses (including the degenerate case of a point), parabolae or hyperbolae (including the degenerate case of a line). The nature of the orbit depends on the energy E and on the relative size of q and p; as we shall see, the details are quite subtle, combining the results from the toy model in Sect. 4.2 with lessons from the role of conic sections as trajectories in the standard Kepler problem.

Focusing on the non-degenerate case $l \neq 0$, we note that the sign of q determines both the direction of the cone (4.128) and the position of the plane (4.134) relative to the origin. If $q > 0$ then the situation is as shown in Fig. 4.3, with the cone in the positive half-space determined by $\vec{J} \cdot \vec{r} > 0$ and the plane (4.134) displaced from the origin in the direction of $\vec{N}$. If $q < 0$ the cone is in the opposite half-space and the plane is displaced from the origin in the direction of $-\vec{N}$. The nature of the intersection between them, however, is independent of the sign of q, and only depends on the angle between $\vec{J}$ and $\vec{N}$, see again Fig. 4.3.

A lengthy calculation shows that the squared norm of $\vec{N}$ is

$$|\vec{N}|^2 = \frac{l^2 E}{4} \left(L^2 E - 2pq \right), \tag{4.135}$$

which is positive for all allowed values of the energy by virtue of (4.115). Since, from the first equation in (4.130),

$$\vec{N} \cdot \vec{J} = \frac{l^2}{2L}(L^2 E - 2q^2 - pq), \tag{4.136}$$

we deduce that δ is determined by

$$\cos \delta = \frac{l}{J} \frac{L^2 E - 2q^2 - pq}{L\sqrt{E(L^2 E - 2pq)}}. \tag{4.137}$$

In order to classify the orbits we also note that, from (4.129) and (4.133), $\sin \theta = \frac{l}{J}$ or

$$\cos \left(\frac{\pi}{2} - \theta \right) = \frac{l}{J}. \tag{4.138}$$

Elementary geometrical considerations in Fig. 4.3 now show that

$$l \neq 0 \quad \text{and} \quad \left\{ \begin{array}{l} \delta < \frac{\pi}{2} - \theta \\ \\ \delta = \frac{\pi}{2} - \theta \\ \\ \frac{\pi}{2} - \theta < \delta < \frac{\pi}{2} + \theta \\ \\ \delta \geq \frac{\pi}{2} + \theta \end{array} \right\} \quad \Leftrightarrow \quad \text{orbit is} \quad \left\{ \begin{array}{l} \text{ellipse} \\ \\ \text{parabola} \\ \\ \text{hyperbola} \\ \\ \text{empty set.} \end{array} \right\} \tag{4.139}$$

We analyse each of those conditions in turn. Since the cosine function is strictly decreasing on the interval $[0, \pi]$, applying it to the inequalities in (4.139) reverses them. It will also be useful to observe that the energy bound (4.115) implies

$$q^2 < \frac{p^2}{4} \Rightarrow L^2 E > 2q^2 + pq. \tag{4.140}$$

To see this one needs to distinguish the cases $pq > 0$ and $pq < 0$. So in the first case we have

$$\begin{aligned} L^2 E &> pq + pq \\ &> 2q^2 + pq, \end{aligned} \tag{4.141}$$

where we have used $|q| < |p/2|$. In the second this condition implies $2q^2 + pq < 0$ and hence the bound $L^2 E > 0$ implies relation (4.140).

For elliptic orbits, we require $\cos \delta > \cos(\frac{\pi}{2} - \theta)$. Inserting the above relations, this condition gives

$$l \neq 0 \quad \text{and} \quad L^2 E - 2q^2 - pq > L\sqrt{E(L^2 E - 2pq)}. \tag{4.142}$$

Since the right hand side is positive (assuming $L > 0$), we deduce that, for elliptic orbits,

$$L^2 E > 2q^2 + pq. \tag{4.143}$$

On the other hand, squaring both sides of (4.142), we deduce

$$L^2 E < \left(q + \frac{p}{2}\right)^2. \tag{4.144}$$

However, the inequalities (4.143) and (4.144) can only both be satisfied if

$$q^2 < \frac{p^2}{4}, \tag{4.145}$$

which is precisely the condition (4.75) derived in the toy model in Sect. 4.2. Since, by (4.140), the condition (4.145) is sufficient for (4.143) to hold, we deduce that elliptical orbits occur iff $p \neq 0$, the charge q satisfy (4.145) and the energy satisfies (4.144) [1]. As an aside we note that elliptical orbits are possible in the case $L < 0$ even when $p = 0$ (as discussed in [3]).

Returning to general p and positive L, the analysis of the conditions (4.139) for the parabolic and hyperbolic cases along the lines of the discussion of elliptical orbits is now straightforward. We skip most details, but point out that, in the hyperbolic case, the trigonometric identity $\cos\left(\frac{\pi}{2}+\theta\right) = -\cos\left(\frac{\pi}{2}-\theta\right)$ applied to (4.139) implies the condition

$$l \neq 0 \quad \text{and} \quad |L^2E - 2q^2 - pq| < L\sqrt{E(L^2E-2pq)}, \tag{4.146}$$

which (for positive L) is equivalent to

$$L^2E > \left(q + \frac{p}{2}\right)^2, \tag{4.147}$$

but does not require any restrictions on p and q.

We summarise the dependence of the orbits on the energy E and the charge q as follows:

$$l \neq 0 \;\text{ and }\; \left\{ \begin{array}{l} p \neq 0, \;\; q^2 < \frac{p^2}{4}, \;\; L^2E < \left(q+\frac{p}{2}\right)^2 \\ p \neq 0, \;\; q^2 \leq \frac{p^2}{4}, \;\; L^2E = \left(q+\frac{p}{2}\right)^2 \\ L^2E > \left(q+\frac{p}{2}\right)^2 \end{array} \right\} \;\Leftrightarrow\; \text{orbit is} \; \left\{ \begin{array}{l} \text{ellipse} \\ \text{parabola} \\ \text{hyperbola.} \end{array} \right\} \tag{4.148}$$

[1] If p were to vanish then (4.145) forces q to vanish, and then (4.144) would imply $E = 0$, which is impossible.

4.5 Gauged Taub-NUT quantum mechanics

4.5.1 Canonical quantisation

In this thesis we set $\hbar = 1$ when discussing quantum mechanics. With this convention, the canonical quantisation procedure of T^*M_{TN} amounts to replacing

$$\vec{\pi} \to -i\frac{\partial}{\partial \vec{r}}, \quad q \to -i\partial_\gamma, \tag{4.149}$$

where $\frac{\partial}{\partial \vec{r}} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$. A comparison of q with (2.72) shows that, as an operator,

$$q = -i\partial_\gamma = -iX_3. \tag{4.150}$$

The relation (4.103) implies the quantisation of the mechanical momentum according to

$$\vec{p} \to -i\frac{\partial}{\partial \vec{r}} + i\vec{A}\partial_\gamma, \tag{4.151}$$

where $\vec{A}$ is the magnetic monopole vector potential (4.97).

Inserting (4.150) into (4.113) gives

$$H_p \;=\; \frac{1}{V}|\vec{p}|^2 - \frac{V}{L^2}\left(X_3 + \frac{ip}{2V}\right)^2, \tag{4.152}$$

which turns out to be precisely the gauged Laplace operator (4.91). To see this, note that

$$\begin{aligned} |\vec{p}|^2 &= (-i\partial_l + i\partial_\gamma A_l)(-i\partial_l + i\partial_\gamma A_l) \\ &= -\left(\frac{\partial}{\partial \vec{r}}\right)^2 + 2\left(\vec{A}\cdot\frac{\partial}{\partial \vec{r}}\right)\partial_\gamma - |\vec{A}|^2\partial_\gamma^2, \end{aligned} \tag{4.153}$$

where $\left(\frac{\partial}{\partial \vec{r}}\right)^2$ is the Laplace operator on Euclidean $\mathbb{R}^3$, and we have used that, for the Dirac monopole (4.97), $\mathrm{div}\vec{A} = 0$. In terms of spherical coordinates and (4.97) one checks that

$$\left(\frac{\partial}{\partial\vec{r}}\right)^2 = \frac{1}{r^2}\partial_r(r^2\partial_r) + \frac{1}{r^2}(\partial_\beta^2 + \cot\beta\partial_\beta + \csc^2\beta\partial_\alpha^2),$$
$$\vec{A}\cdot\frac{\partial}{\partial\vec{r}} = \frac{\cos\beta}{r^2\sin^2\beta}\partial_\alpha, \quad |\vec{A}|^2 = \frac{\cos^2\beta}{r^2\sin^2\beta}, \tag{4.154}$$

so that

$$|\vec{p}|^2 = -\frac{1}{r^2}\partial_r(r^2\partial_r) - \frac{1}{r^2}(\partial_\beta^2 + \cot\beta\partial_\beta + \csc^2\beta\partial_\alpha^2 - 2\cot\beta\csc\beta\partial_\gamma\partial_\alpha + \cot^2\beta\partial_\gamma^2)$$
$$= -\frac{1}{r^2}\partial_r(r^2\partial_r) - \frac{1}{r^2}(X_1^2 + X_2^2), \tag{4.155}$$

where we have used the relation,

$$X_1^2 + X_2^2 = \partial_\beta^2 + \cot\beta\partial_\beta + \cot^2\beta\partial_\gamma^2 + \csc^2\beta\partial_\alpha^2 - 2\cot\beta\csc\beta\partial_\alpha\partial_\gamma, \tag{4.156}$$

which can be obtained from (3.80). Substituting (4.155) into (4.152) shows that the quantum Hamiltonian H_p is the gauged Laplace operator (4.91), as claimed.

Finally applying the quantisation rule to the angular momentum $\vec{J}$ defined in (4.116) we obtain the differential operator

$$\vec{J} = -i\vec{r}\times\frac{\partial}{\partial\vec{r}} + i(\vec{r}\times\vec{A} - \hat{r})\partial_\gamma$$
$$= i\begin{pmatrix} \sin\alpha\,\partial_\beta + \cot\beta\cos\alpha\,\partial_\alpha - \frac{\cos\alpha}{\sin\beta}\partial_\gamma \\ -\cos\alpha\,\partial_\beta + \cot\beta\sin\alpha\,\partial_\alpha - \frac{\sin\alpha}{\sin\beta}\partial_\gamma \\ -\partial_\alpha \end{pmatrix}. \tag{4.157}$$

One checks that, up a factor of i, the components are the vector fields Z_1, Z_2 and Z_3 (2.85) generating the left-action of $SU(2)$ on itself:

$$\vec{J} = i\vec{Z}. \tag{4.158}$$

Observe from (2.87) that the squared total angular momentum operator can be written in terms of the left- and right-generated vector fields on S^3 as

$$\vec{J}^2 = -(Z_1^2 + Z_2^2 + Z_3^2) = -(X_1^2 + X_2^2 + X_3^2) = -\Delta_{S^3}. \tag{4.159}$$

4.5.2 Separating variables

For fixed r, the angular part of the quantum Hamiltonian (4.91) is akin to the Hamiltonian of a symmetric rigid body coupled to a gauge field. In that context, the operators iZ_j are interpreted as 'space-fixed' angular momentum components and the operators iX_j as 'body-fixed' angular momentum components [3]. The quantum Hamiltonian H_p commutes with Z_1, Z_2, Z_3 and with X_3; together, these generate the $U(2)$ symmetry of TN space.

To separate the radial from the angular dependence in the wavefunction, we therefore require a complete set of functions on $SU(2)$ which diagonalise the commuting operators Δ_{S^3}, iZ_3, iX_3. This is usually done in terms of Wigner functions of the Euler angles, but here we use the construction of the eigenfunctions as homogeneous polynomials of the complex coordinates z_1, z_2 and their complex conjugates given in (2.102). These functions are normalised and are clearly orthogonal since they are eigenfunctions of the Hermitian operators Δ_{S^3}, (total angular momentum), iZ_3 (angular momentum along the space-fixed 3-axis), iX_3 (angular momentum along the body-fixed 3-axis) with eigenvalues given in (2.104). They also satisfy (2.105) which shows that all the angular momentum eigenstates can be obtained from the holomorphic Y^j_{jm} or the anti-holomorphic Y^j_{-jm} by the repeated action of X_- or X_+.

We look for stationary states Ψ of the form

$$\Psi(r, z_1, z_2) = R(r) Y^j_{sm}(z_1, z_2). \tag{4.160}$$

Using

$$(X_1^2 + X_2^2) Y^j_{sm} = [-j(j+1) + s^2] Y^j_{sm} \tag{4.161}$$

in (4.91) then the stationary Schrödinger equation

$$H_p\Psi = E\Psi, \tag{4.162}$$

gives the radial equation

$$\left[-\frac{1}{r^2}\partial_r(r^2\partial_r) + \frac{1}{r^2}j(j+1) + \left(\frac{2s^2}{L} - \frac{ps}{L} - EL\right)\frac{1}{r} + \left(\left(\frac{s-\frac{p}{2}}{L}\right)^2 - E\right)\right]R(r) = 0. \tag{4.163}$$

Before we study bound and scattering states in the following sections, we make two general observations.

It follows from (4.150) that, when acting on the functions (2.102), the operator q has the eigenvalue $-s$. For later use, note that the classical bound (4.115) also holds in the quantum case, so that, in particular, for any eigenstate of H_p and q with eigenvalues E and $-s$, we have

$$L^2E \geq \begin{cases} 0 & \text{if } ps > 0 \\ -2sp & \text{if } ps \leq 0. \end{cases} \tag{4.164}$$

Finally, it is worth stressing that neither the space-fixed nor the body-fixed angular momentum operators discussed above are invariant under $U(1)$-gauge transformations. The quantum numbers j, s and m are not gauge invariant either and therefore have to be interpreted with care. However, this is familiar in the context of the Schrödinger equation coupled to a magnetic field, particularly in the discussion of Landau levels for planar motion in a magnetic field. Even though the angular momentum operator is not gauge invariant in this context, the eigenvalues can be used to label degenerate energy eigenstates. This labelling is not gauge invariant, but physical quantities like the energy or the degeneracy of energy levels are. The role of the gauge choice in labelling degenerate states in Landau levels is discussed in detail in [42], see also the book [11].

4.5.3 Bound states

The substitution of

$$R(r) = r^j e^{-k'r} u(r), \quad k'^2 = \left(\frac{s - \frac{p}{2}}{L} \right)^2 - E, \tag{4.165}$$

into the radial Schrödinger equation (4.163), reduces it to

$$z \frac{d^2u}{dz^2} + (b - z) \frac{du}{dz} - au(z) = 0, \tag{4.166}$$

where

$$z = 2k'r, \quad a = (j + 1 + \lambda), \;\; b = 2j + 2, \tag{4.167}$$

and

$$\lambda = -\frac{1}{2k'L} \left(L^2 E + ps - 2s^2 \right). \tag{4.168}$$

The equation (4.166) is the confluent hypergeometric equation [6]. The general solution which is regular at the origin is

$$u = A r^j e^{-k'r} M(a, b, z), \tag{4.169}$$

where A is an arbitrary constant and M is Kummer's function of the first kind. Square integrability requires

$$a = -\nu, \qquad \nu = 0, 1, 2, \ldots \quad \text{and} \quad k' \in \mathbb{R}^+. \tag{4.170}$$

Since j takes arbitrary half-integer positive values, the first condition is equivalent to

$$n := -\lambda = \nu + j + 1, \qquad \nu = 0, 1, 2, \ldots. \tag{4.171}$$

In principle, n can take all half-integer values ≥ 1, but the ranges of the quantum numbers j and n are related by

$$n = j + 1 + \nu, \qquad \nu = 0, 1, 2, \ldots. \tag{4.172}$$

This requirement together with the expression (4.168) for λ as well as $L > 0$ and

$k' > 0$ imply

$$L^2 E > 2s^2 - ps. \tag{4.173}$$

On the other hand, the relation (4.165) between E and k' enforces

$$L^2 E < \left(s - \frac{p}{2}\right)^2 = s^2 - ps + \frac{p^2}{4}. \tag{4.174}$$

There can only be bound states if these two inequalities can be simultaneously satisfied, i.e., if

$$s^2 < \frac{p^2}{4}, \tag{4.175}$$

which is the quantum version of the condition (4.145) for bounded orbits in the classical theory.

Note that, if L were negative, the inequality (4.173) would have the opposite direction and there would be no condition on p. In that case we can set $p = 0$ and recover the bound states in the singular $L = -2$ TN space discussed in [3], which exist for any $s \neq 0$. As shown in [43], there are no bound states (and no bounded orbits) when $L > 0$ and $p = 0$. More generally, however, binding is always possible when p is sufficiently large. All these results confirm the qualitative discussion of the 2-dimensional toy model in Sect. 4.2.

Solving (4.165) and (4.171) for E, we find

$$E = \frac{2}{L^2}\left[-n^2 + s^2 - \frac{ps}{2}\right] \pm \frac{2n}{L^2}\sqrt{n^2 - s^2 + \frac{p^2}{4}}. \tag{4.176}$$

Only the solution with the upper sign satisfies (4.173), and we write the resulting spectrum of bound state energies as

$$E = \frac{2}{L^2}\left[s^2 - \frac{ps}{2} + n\sqrt{n^2 - s^2 + \frac{p^2}{4}} - n^2\right], \qquad n = |s|+1, |s|+2, |s|+3\ldots. \tag{4.177}$$

The behaviour for large n is typical for Coulomb bound states

$$E \approx \frac{\left(s - \frac{p}{2}\right)^2}{L^2} - \frac{\left(s^2 - \frac{p^2}{4}\right)^2}{4L^2 n^2} + \mathcal{O}\left(\frac{1}{n^4}\right). \tag{4.178}$$

This formula for the bound state energy shows that, like in the toy model of Sect. 4.2, the bound state energies are relatively high when p and q have the same sign (so that the signs of p and s are opposite) but are lowered when the signs of p and q are opposite (and those of p and s the same). Note also that, in the limit $p = 0$ and for $L = -2$ our formula reduces to that obtained in [3] for the negative mass TN space. For a detailed comparison observe that in [3] only integer values of j and n were considered.

4.5.4 Degeneracy

The energy levels for fixed s and n have a large degeneracy, given by the sum over the dimension $2j+1$ for allowed values of j. We can compute the degeneracy of the energy spectrum by observing that the inequality $j \geq s$ together with (4.172) implies the condition $s \leq j \leq n-1$. Then the degeneracy is given by the sum

$$\sum_{j=s}^{n-1} 2j+1. \tag{4.179}$$

In the case when j is an integer this sum can be computed directly,

$$\begin{aligned} \sum_{j=s}^{n-1} 2j+1 &= \sum_{j=s}^{n-1} 2j + (n-1-s) + 1 \\ &= 2\sum_{j=1}^{n-1} j - 2\sum_{j=1}^{s-1} j + n - s \\ &= n^2 - s^2. \end{aligned} \tag{4.180}$$

Now if j is half an integer we can write

$$j = \frac{2k-1}{2}, \quad k = 1, 2, 3, \ldots. \tag{4.181}$$

So in terms of the new variable k the above sum reads

$$
\begin{aligned}
\sum_{k=s+\frac{1}{2}}^{n-\frac{1}{2}} 2k &= 2\sum_{k=1}^{n-\frac{1}{2}} k - 2\sum_{k=1}^{s-\frac{1}{2}} k \\
&= (n-\frac{1}{2})(n+\frac{1}{2}) - (s-\frac{1}{2})(s+\frac{1}{2}) \\
&= n^2 - s^2. \qquad (4.182)
\end{aligned}
$$

As we shall see in Sect. 4.6, the degeneracy can be understood in terms of the Runge-Lenz vector.

4.5.5 Scattering states

Next we turn to solutions of the eigenvalue equation (4.162) which describe stationary scattering states. For the analysis of scattering it is convenient to use parabolic coordinates familiar from the treatment of Coulomb scattering.

Assuming solutions of (4.162) of the form

$$\Psi = e^{-is\gamma} e^{im\alpha} \Lambda(\beta, r), \tag{4.183}$$

and recalling the formula (4.156) we find that Λ has to satisfy the equation

$$
\begin{aligned}
\left(\partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\partial_\beta^2 + \frac{1}{r^2}\cot\beta\,\partial_\beta\right)\Lambda + \frac{1}{r^2}(-s^2\cot^2\beta - m^2\csc^2\beta + 2ms\cot\beta\csc\beta)\Lambda \\
-\frac{V^2}{L^2}\left(s - \frac{p}{2V}\right)^2\Lambda + EV\Lambda = 0.
\end{aligned}
\tag{4.184}
$$

Now introducing parabolic coordinates ξ, η via

$$\xi = r(1+\cos\beta), \qquad \eta = r(1-\cos\beta), \tag{4.185}$$

and noting the inverse transformation

$$r = \frac{\xi+\eta}{2}, \quad \cos\beta = \frac{\xi-\eta}{\xi+\eta}, \quad \sin\beta = \frac{2\sqrt{\xi\eta}}{\xi+\eta}, \tag{4.186}$$

we think of Λ now as a function of ξ and η via the substitution (4.186) for r and β. Then (4.184) becomes

$$\frac{4}{\xi+\eta}\left(\xi\partial_\xi^2+\eta\partial_\eta^2+\partial_\xi+\partial_\eta\right)\Lambda-\frac{1}{\xi\eta}\left(s^2+m^2+2ms\frac{\xi-\eta}{\xi+\eta}\right)\Lambda$$
$$+\left(-\frac{2s^2}{L}+\frac{sp}{L}+EL\right)\frac{2\Lambda}{\xi+\eta}-\frac{1}{L^2}\left(s-\frac{p}{2}\right)^2\Lambda+E\Lambda=0. \tag{4.187}$$

Separating variables again via $\Lambda=f(\xi)g(\eta)$ we find

$$\frac{4\xi}{f}\partial_\xi^2 f+\frac{4}{f}\partial_\xi f-\frac{1}{\xi}(m+s)^2+k^2\xi+2\left(EL-\frac{2s^2}{L}+\frac{ps}{L}\right)-C=0, \tag{4.188}$$

$$\frac{4\eta}{g}\partial_\eta^2 g+\frac{4}{g}\partial_\eta g-\frac{1}{\eta}(m-s)^2+k^2\eta+C=0, \tag{4.189}$$

where C is a separation constant and

$$k^2=E-\frac{1}{L^2}\left(s-\frac{p}{2}\right)^2. \tag{4.190}$$

These differential equations can be simplified further if we assume solutions of the form

$$f(\xi)=\xi^{\frac{|m-s|}{2}}e^{-\frac{ik\xi}{2}}F(\xi),\quad g(\eta)=\eta^{\frac{|m+s|}{2}}e^{-\frac{ik\eta}{2}}G(\eta), \tag{4.191}$$

and implement the change of variable

$$z_1=ik\xi,\qquad z_2=ik\eta. \tag{4.192}$$

Doing so we deduce that both F and G satisfy the confluent hypergeometric equation

$$z_1\frac{d^2F}{dz_1^2}+(b_1-z_1)\frac{dF}{dz_1}-a_1F=0, \tag{4.193}$$

$$z_2\frac{d^2G}{dz_2^2}+(b_2-z_2)\frac{dG}{dz_1}-a_2G=0, \tag{4.194}$$

where

$$a_1=\frac{|m-s|}{2}+\frac{1}{2}-\frac{ic}{4k}+\frac{i}{2k}\left(EL-\frac{2s^2}{L}+\frac{ps}{L}\right),\quad b_1=|m-s|+1, \tag{4.195}$$

$$a_2 = \frac{|m+s|}{2} + \frac{1}{2} + \frac{ic}{4k}, \quad b_2 = |m+s| + 1. \tag{4.196}$$

We see from these relations that

$$a_1 + a_2 = 1 + \frac{1}{2}|m+s| + \frac{1}{2}|m-s| - i\lambda, \tag{4.197}$$

where

$$\lambda = -\frac{L}{2kL^2}\left(L^2 E + ps - 2s^2\right). \tag{4.198}$$

So, the scattering solution is of the form

$$\Psi = e^{-is\gamma} e^{im\alpha} \xi^{\frac{|m-s|}{2}} \eta^{\frac{|m+s|}{2}} e^{-\frac{ik\xi}{2}} e^{-\frac{ik\eta}{2}} M(a_1, b_1, ik\xi) M(a_2, b_2, ik\eta), \tag{4.199}$$

where M is Kummer's function of the first kind. This is formally the same as the two monopole scattering state found by Gibbons and Manton [3] in the case $p = 0$ and $L = -2$. However, in our case the constants k and λ have an extra p-dependence and the length parameter L is positive.

As in [3] one can compute the cross section by looking at the wave function with $m = s$ and $a_1 = 1$,

$$\Psi = e^{is(\alpha-\gamma)}(r-z)^{|s|} e^{ikz} M(|s| - i\lambda, 2|s|+1, ik(r-z)), \quad z = r\cos\beta. \tag{4.200}$$

Then the substitution of the asymptotic form of $M(|s| - i\lambda, 2|s|+1, ik(r-z))$ for large $|z|$ as in [3] allows us to identify the scattered spherical wave, and to obtain the cross section

$$\frac{d\sigma}{d\Omega} = \frac{s^2 + \lambda^2}{4k^2} \csc^4 \tfrac{\beta}{2}. \tag{4.201}$$

Writing λ in terms of k via (4.198), we finally arrive at the cross section

$$\begin{aligned} \left(\frac{d\sigma}{d\Omega}\right)_{(p,s)} &= \frac{L^2}{16}\left[\frac{4s^2}{k^2L^2} + \left(1 - \frac{s^2}{k^2L^2} + \frac{p^2}{4k^2L^2}\right)^2\right] \csc^4 \tfrac{\beta}{2} \\ &= \frac{L^2}{16}\left[\left(1 + \frac{s^2}{k^2L^2}\right)^2 + \frac{p^2}{4k^2L^2}\left(2 - \frac{2s^2}{k^2L^2} + \frac{p^2}{4k^2L^2}\right)\right] \csc^4 \tfrac{\beta}{2}. \end{aligned} \tag{4.202}$$

The special cases $s = 0$ and $p = 0$ are interesting because the resulting cross sections

$$\left(\frac{d\sigma}{d\Omega}\right)_{(p,0)} = \frac{L^2}{16}\left(1 + \frac{(p/2)^2}{L^2k^2}\right)^2 \csc^4\tfrac{\beta}{2} \tag{4.203}$$

and

$$\left(\frac{d\sigma}{d\Omega}\right)_{(0,s)} = \frac{L^2}{16}\left(1 + \frac{s^2}{L^2k^2}\right)^2 \csc^4\tfrac{\beta}{2} \tag{4.204}$$

are mapped into each other under the exchange $s \leftrightarrow p/2$ even though the general case (4.202) is not invariant under this exchange. In both special cases, the charge, energy and angular dependence approaches that of the Rutherford scattering cross section for electrically charged particles in the limit of large s (or p).

4.6 Algebraic calculation of quantum bound states

4.6.1 The Runge-Lenz operator and *so*(4) symmetry

In 1926, Pauli computed the quantum spectrum of the Kepler problem by using the conservation of the Runge-Lenz vector [44]. His method has since then been much explored and extended in various papers, see [45] for a reference which is particularly useful in the current context. More recently, it was used to compute bound states and scattering of the Laplace operator on TN space [41] and also for the Dirac operator on TN [28]. We now use it to re-derive the spectrum of the gauged TN Hamiltonian (4.91) purely algebraically.

As always in quantising a theory, we need to be careful with ordering in the quantisation of classically conserved quantities. While there are no such ambiguities in the definition of the angular momentum operator, they do arise in defining a quantum version of the Runge-Lenz vector. The quantum analogues of the canonical Poisson brackets (4.111),

$$[p_i, p_j] = -iq\epsilon_{ijk}\frac{x_k}{r^3}, \quad [p_j, f(\vec{r})] = -i\partial_j f(\vec{r}), \tag{4.205}$$

imply, for the quantum angular momentum operator (4.157),

$$[J_i, p_j] = i\epsilon_{ijk} p_k, \tag{4.206}$$

which is the quantum version of (4.117). This means that $[J_i, p_j] \neq 0,\ i \neq j$ and hence the order of $\vec{J}$ and $\vec{p}$ is important in the definition of the quantum version of the Runge-Lenz vector (4.123).

Noting that, classically, $\vec{p} \times \vec{J} = \frac{1}{2}(\vec{p} \times \vec{J} - \vec{J} \times \vec{p})$, one finds that the quantum ordering

$$\vec{M} = \frac{1}{2}(\vec{p} \times \vec{J} - \vec{J} \times \vec{p}) - \frac{\hat{r}}{2L}\left(L^2 H - 2q^2\right) \tag{4.207}$$

ensures that the quantum commutation relations between $\vec{J}$ and $\vec{M}$ are

$$[J_k, J_l] = i\epsilon_{klm} J_m, \quad [J_k, M_l] = i\epsilon_{klm} M_m, \tag{4.208}$$

in analogy to the classical Poisson brackets (4.118) and (4.120) respectively. Now we use (4.206) to rewrite the Runge-Lenz vector as

$$\vec{M} = \vec{p} \times \vec{J} - i\vec{p} - \frac{\hat{r}}{2L}\left(L^2 H - 2q^2\right). \tag{4.209}$$

The second ambiguity has to do with the position of the factor $\hat{r}$ of the last term. The above choice guarantees that the quantum Runge-Lenz vector commutes with H.

In order to obtain a Runge-Lenz vector which commutes with the gauged Hamiltonian H_p and still satisfies the relations (4.208), it turns out that the addition of the term $\vec{f}$ (4.122), which worked in the classical case, also works in the quantum theory. The gauged quantum Runge-Lenz vector is therefore

$$\vec{M}^p = \vec{p} \times \vec{J} - i\vec{p} - \frac{\hat{r}}{2L}\left(L^2 H_p - 2q^2 - pq\right). \tag{4.210}$$

A lengthy calculation yields the commutators

$$[J_i, M_j^p] = i\epsilon_{ijk} M_k^p,$$
$$[M_i^p, M_j^p] = i\left[\frac{1}{L^2}\left(q + \frac{p}{2}\right)^2 - H_p\right]\epsilon_{ijk} J_k, \tag{4.211}$$

which quantise the Poisson brackets (4.124). See Appendix A.1 for a derivation of the second commutator. The first commutator can be easily deduced from (4.208) and the relation $\vec{M}^p = \vec{M} - \vec{f}$ where $\vec{f}$ is given in (4.122). See also Appendices A.2 and A.3 for a proof of the following operator identities:

$$\vec{M}^p \cdot \vec{J} = \vec{J} \cdot \vec{M}^p = -\frac{q}{2L}\left(L^2 H_p - 2q^2 - pq\right),$$
$$\vec{M}^p \cdot \vec{M}^p = \left[H_p - \frac{1}{L^2}\left(q + \frac{p}{2}\right)^2\right](\vec{J}\cdot\vec{J} - q^2 + 1) + \frac{1}{4L^2}\left(L^2 H_p - 2q^2 - pq\right)^2. \tag{4.212}$$

Since the Hamiltonian H_p and the $U(1)$ generator q commute with each other and all components of $\vec{M}^p$ and $\vec{J}$, we can fix their eigenvalues and study the commutation relations of $\vec{M}^p$ and $\vec{J}$ in a fixed common eigenspace of H_p and q. Denoting the eigenvalues by, respectively, E and $-s$, and assuming the bound state energy range

$$L^2 E < \left(s - \frac{p}{2}\right)^2, \tag{4.213}$$

we define the rescaled Runge-Lenz vector

$$\tilde{M}^p = \frac{1}{\sqrt{\frac{1}{L^2}(s - \frac{p}{2})^2 - E}}\vec{M}^p. \tag{4.214}$$

Together with the components of $\vec{J}$, it satisfies the $so(4)$ commutation relations

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \quad [J_i, \tilde{M}_j^p] = i\epsilon_{ijk}\tilde{M}_k^p, \quad [\tilde{M}_i^p, \tilde{M}_j^p] = i\epsilon_{ijk} J_k. \tag{4.215}$$

4.6.2 Bound states revisited

The bound state energies of H_p can now be derived from the isomorphism $so(4) \simeq su(2) \oplus su(2)$ and the standard representation theory of $su(2)$. We introduce the commuting operators

$$\vec{J}_{\pm} = \frac{1}{2}(\vec{J} \pm \tilde{M}^p), \tag{4.216}$$

and see that the two Casimirs

$$J_{\pm}^2 = \frac{1}{4}(\vec{J}\cdot\vec{J} + \tilde{M}^p\cdot\tilde{M}^p) \pm \frac{1}{4}(\tilde{M}^p\cdot\vec{J} + \vec{J}\cdot\tilde{M}^p) \tag{4.217}$$

have eigenvalues $j_{\pm}(j_{\pm}+1)$, where $j_{\pm}$ are both non-negative half-integers. Moreover, since $\vec{J} = \vec{J}_+ + \vec{J}_-$, it follows that the total angular momentum quantum number j defined in (2.104) lies in the range

$$|j_+ - j_-| \leq j \leq |j_+ + j_-|. \tag{4.218}$$

Since $-j \leq s \leq j$, we deduce that

$$-|j_+ + j_-| \leq s \leq |j_+ + j_-|. \tag{4.219}$$

In terms of $\tilde{M}^p$, the relations (4.212) read

$$\begin{aligned} \tilde{M}^p\cdot\vec{J} = \vec{J}\cdot\tilde{M}^p &= \frac{s}{2L}\frac{(L^2E - 2s^2 + ps)}{\sqrt{\frac{1}{L^2}(s-\frac{p}{2})^2 - E}}, \\ \tilde{M}^p\cdot\tilde{M}^p + \vec{J}\cdot\vec{J} &= s^2 - 1 + \frac{(L^2E - 2s^2 + ps)^2}{4((s-\frac{p}{2})^2 - L^2E)}. \end{aligned} \tag{4.220}$$

Substituting these into (4.217) and replacing $\vec{J}_{\pm}^2$ by the eigenvalues $j_{\pm}(j_{\pm}+1)$, we get two quadratic equations for the unknown[2]

$$n := \frac{L^2E - 2s^2 + ps}{2L\sqrt{\frac{1}{L^2}(s-\frac{p}{2})^2 - E}}, \tag{4.221}$$

namely

$$\begin{aligned} n^2 + 2sn + s^2 - 1 - 4j_+(j_+ + 1) &= 0, \\ n^2 - 2sn + s^2 - 1 - 4j_-(j_- + 1) &= 0. \end{aligned} \tag{4.222}$$

[2]Note that the definition of n here is consistent with (4.168) and (4.171)

The roots of the first equation are

$$n = -s \pm (2j_+ + 1), \tag{4.223}$$

and the roots for the second equation are

$$n = s \pm (2j_- + 1). \tag{4.224}$$

Both equations have to be satisfied for some values of j_+ and j_-, but combining the upper sign in one with the lower sign in the other implies a value of s which is outside the range (4.219). Hence, there are only two possible solutions for n, one which is manifestly a half-integer ≥ 1

$$n = -s + (2j_+ + 1) = s + (2j_- + 1) \tag{4.225}$$

and one which is manifestly a half-integer ≤ -1

$$n = -s - (2j_+ + 1) = s - (2j_- + 1). \tag{4.226}$$

Finally solving (4.221) for E we obtain again the solutions (4.176) previously obtained via square integrability arguments. However, we still have four possibilities in total: two choices of sign in (4.176) and two choices for n (positive or negative) and in this section we cannot assume the conditions (4.173) and (4.175) to resolve the ambiguity. In the four different cases the energy equation (4.176) takes the two possible forms

$$\frac{L^2 E}{2} - s^2 + \frac{ps}{2} = n^2 \left(\pm \sqrt{1 + \frac{\frac{p^2}{4} - s^2}{n^2}} - 1 \right). \tag{4.227}$$

We can eliminate the lower sign because it conflicts with the lower bound (4.164). To see this we re-write (4.227) with the lower sign as

$$L^2 E = -ps - 2(n^2 - s^2) - 2n^2 \sqrt{1 + \frac{\frac{p^2}{4} - s^2}{n^2}}, \tag{4.228}$$

showing that $L^2E < -ps$ in this case. However, this is inconsistent with the energy inequality (4.164) and therefore ruled out.

Turning to the upper sign, we need to consider the two possible signs of n and check the consistency between (4.227) and (4.221). In the case $n \geq 1$, both (4.227) and (4.221) assign a positive sign to $\frac{L^2E}{2} - s^2 + \frac{ps}{2}$ provided $s^2 < p^2/4$. In that case we arrive at the previously derived energy spectrum (4.177) together with the condition (4.175) for bound states. However, $n \leq -1$ is also consistent provided $s^2 > p^2/4$. We have not been able to eliminate this case using only the algebraic methods of this section. It seems that the consideration of the actual wavefunction (4.162) and integrability requirement (4.171) is needed to rule out $n \leq -1$.

Finally turning to the degeneracy of the energy levels, we see that the quantum numbers n and s are determined via

$$n = j_+ + j_- + 1, \quad s = j_+ - j_-, \tag{4.229}$$

and that the degeneracy of the energy level with quantum numbers n and s is the dimension of the tensor product $J_+ \otimes J_-$ of the irreducible angular representations with spins j_+ and j_-,

$$(2j_+ + 1)(2j_- + 1) = n^2 - s^2, \tag{4.230}$$

reproducing and interpreting the degeneracy (4.182) of energy levels.

In this section we have only studied the commutation relations of the angular momentum and Runge-Lenz vectors at energies satisfying $L^2E < (s - p/2)^2$ and corresponding to bound states. It is not difficult to modify our discussion for the case $L^2E \geq (s-p/2)^2$. The dynamical symmetry algebra of the angular momentum and of a suitable rescaled Runge-Lenz vector, analogous to (4.215), turns out to be isomorphic to the Lie algebra $so(3) \ltimes \mathbb{R}^3$ of the Euclidean group when $L^2E = (s - p/2)^2$ and isomorphic to the Lie algebra $so(3,1)$ of the Lorentz group when $L^2E > (s - p/2)^2$. In the following section we will see how these three cases can be understood from a unified, geometrical point of view.

4.7 Twistorial derivation of the gauged Runge-Lenz vector

4.7.1 Twistors, $SU(2,2)$ symmetry and moment maps

It has been known for a while [12] that the usual Kepler problem can be regularised by embedding momentum 3-space into the 3-sphere by means of a stereographic projection. This gives a geometrical picture of the angular momentum and Runge-Lenz vectors as conserved quantities associated with symmetries of the round 3-sphere. For a full geometrical understanding of the dynamical symmetry of the Kepler problem, it is moreover convenient to think of it as the symplectic quotient of an 8-dimensional phase space, and the dynamical symmetry algebra for the various energy regimes as subalgebras of $so(4,2)$, see [37] for a pedagogical review.

It was shown in [46] that one can similarly interpret angular momentum and Runge-Lenz vectors of the (ungauged) TN motion as generators of a subalgebra of an $su(2,2) \simeq so(4,2)$ symmetry algebra acting on an 8-dimensional phase space of twistors. In this section we show how this story can be extended to the gauged TN dynamics. We begin with a brief review of the relevant notation.

For our purposes, twistor space is $\mathbb{T} = (\mathbb{C}^2 \times \mathbb{C}^2)\backslash\{0\}$, and a twistor

$$Z^\alpha = \begin{pmatrix} \omega \\ \pi \end{pmatrix} \in \mathbb{T} \tag{4.231}$$

is a pair of spinors $\omega = \binom{\omega_1}{\omega_2}$ and $\pi = \binom{\pi_1}{\pi_2}$. The conjugate of Z^α is written as a row vector with lower index Z^*_α whose components are given by

$$Z^*_\alpha = h_{\alpha\beta}\bar{Z}^\beta, \quad h = \begin{pmatrix} 0 & \tau_0 \\ \tau_0 & 0 \end{pmatrix}, \tag{4.232}$$

where $\bar{Z}^\beta$ are the complex conjugates of the components Z^β and we write again τ_0 for the 2×2 identity matrix. The metric $h_{\alpha\beta}$, which lower the index, has signature $(2,2)$ as the matrix h has the eigenvalues $(1,1,-1,-1)$. Twistor space $\mathbb{T}$ is endowed with a pairing

$$Z^*_\alpha Z^\alpha = \bar{\pi}_1\omega_1 + \bar{\pi}_2\omega_2 + \bar{\omega}_1\pi_1 + \bar{\omega}_2\pi_2, \tag{4.233}$$

which can be written as the matrix product Z^*Z where

$$Z^* = Z^\dagger h. \tag{4.234}$$

The pairing is invariant under $U(2,2)$, but we are particularly interested in the Lie algebra of the subgroup $SU(2,2)$. We pick generators γ_{KL} where the structure constants are constant [3], and again write τ_i for the Pauli matrices:

$$\gamma_{0k} = -\frac{1}{2}\begin{pmatrix} \tau_k & 0 \\ 0 & -\tau_k \end{pmatrix}, \quad \gamma_{ij} = -\frac{i}{2}\epsilon_{ijk}\begin{pmatrix} \tau_k & 0 \\ 0 & \tau_k \end{pmatrix}, \quad \gamma_{06} = -\frac{i}{2}\begin{pmatrix} 0 & \tau_0 \\ \tau_0 & 0 \end{pmatrix},$$
$$\gamma_{k6} = -\frac{i}{2}\begin{pmatrix} 0 & \tau_k \\ -\tau_k & 0 \end{pmatrix}, \quad \gamma_{05} = -\frac{i}{2}\begin{pmatrix} 0 & \tau_0 \\ -\tau_0 & 0 \end{pmatrix}, \quad \gamma_{k5} = -\frac{i}{2}\begin{pmatrix} 0 & \tau_k \\ \tau_k & 0 \end{pmatrix},$$
$$\gamma_{56} = \frac{1}{2}\begin{pmatrix} \tau_0 & 0 \\ 0 & -\tau_0 \end{pmatrix}, \quad (\gamma_{LK} = -\gamma_{KL}, \quad K, L = 0, \ldots, 3, 5, 6;\; i, j, k = 1, 2, 3). \tag{4.235}$$

For us, two sub-Lie algebras will be important. The stabiliser Lie algebra of the generator γ_{06} is the Lie algebra generated by $s_i := \frac{1}{2}\epsilon_{ijk}\gamma_{jk}$ and $t_i := \gamma_{5i}$ and is isomorphic to $so(4)$:

$$[s_i, s_j] = \epsilon_{ijk}s_k, \quad [s_i, t_j] = \epsilon_{ijk}t_k, \quad [t_i, t_j] = \epsilon_{ijk}s_k. \tag{4.236}$$

The stabiliser Lie algebra of the generator γ_{05} is the Lie algebra generated by s_i and $r_i := \gamma_{i6}$ and is isomorphic to $so(3,1)$:

$$[s_i, s_j] = \epsilon_{ijk}s_k, \quad [s_i, r_j] = \epsilon_{ijk}r_k, \quad [r_i, r_j] = -\epsilon_{ijk}s_k. \tag{4.237}$$

[3] This is slightly different to the convention in [46] where the structure constants are purely imaginary.

As discussed in [46][4], the space $\mathbb{T}$ has the $U(2,2)$ invariant 1-form

$$\theta = \mathrm{Im}(Z^* dZ) = \frac{1}{2i}(Z^*_\alpha dZ^\alpha - Z^\alpha dZ^*_\alpha), \tag{4.238}$$

whose exterior derivative

$$\Omega = d\theta = -idZ^* \wedge dZ \tag{4.239}$$

is a symplectic form on $\mathbb{T}$. The $su(2,2)$ generators defines a vector field $X_{\gamma_{KL}}$ which generates the infinitesimal action on Z,

$$X_{\gamma_{KL}} : Z \mapsto \gamma_{KL} Z, \tag{4.240}$$

and an action on Z^* which follows from the previous action and (4.234),

$$X_{\gamma_{KL}} : Z^* \to (h\gamma_{KL}Z)^\dagger = Z^\dagger(\gamma_{KL})^\dagger h^\dagger = Z^\dagger hh\gamma_{KL}h = -Z^*\gamma_{KL}. \tag{4.241}$$

where we have used the fact that $h\gamma_{KL}h = -\gamma_{KL}$. Using indices we can rewrite this action as

$$X_{\gamma_{KL}} : Z^\alpha \to (\gamma_{KL})^\alpha_\beta Z^\beta, \quad X_J : Z^*_\alpha \to -Z^*_\beta(\gamma_{KL})^\beta_\alpha. \tag{4.242}$$

Then we see that

$$X_{\gamma_{KL}} = (\gamma_{KL})^l_\beta Z^\beta \frac{\partial}{\partial Z^l} - Z^*_\beta(\gamma_{KL})^\beta_l \frac{\partial}{\partial Z^*_l}. \tag{4.243}$$

Finally evaluating the symplectic form $\Omega = -idZ^*_\alpha \wedge dZ^\alpha$ (4.239) at this vector field we get

$$\begin{aligned} \Omega(X_{\gamma_{KL}},\ ) &= i\delta^\alpha_l dZ^*_\alpha(\gamma_{KL})^l_\beta Z^\beta + i\delta^l_\alpha Z^*_\beta(\gamma_{KL})^\beta_l dZ^\alpha \\ &= i[dZ^*_l(\gamma_{KL})^l_\beta Z^\beta + Z^*_\beta(\gamma_{KL})^\beta_l dZ^l] \\ &= id[Z^*_l(\gamma_{KL})^l_\beta Z^\beta], \end{aligned} \tag{4.244}$$

and so the moment maps are

$$J_{KL} = Z^*\gamma_{KL}Z. \tag{4.245}$$

[4]Note that our sign conventions differ from those in [46]

The diagonal $U(1)$ subgroup of $U(2,2)$ acts on $\mathbb{T}$, preserving its symplectic structure. The moment map is $\frac{1}{2}Z^*Z$, and the symplectic quotient is the level set

$$\mathbb{T}_q = \left\{ Z \in \mathbb{T} \,\middle|\, \frac{1}{2}Z^*Z = q \right\} \tag{4.246}$$

quotiented by the diagonal $U(1)$ action:

$$\tilde{\mathcal{M}}_q = \mathbb{T}_q/U(1) = \mathbb{CP}^3. \tag{4.247}$$

We now introduce coordinates on $\mathbb{T}$ which are particularly well adapted for describing this quotient. With the notation $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$, we parametrise the spinors π and ω in terms of spherical coordinates $(R, \alpha, \beta, \gamma)$ and $\vec{P} \in \mathbb{R}^3, q \in \mathbb{R}$ as

$$\pi = \sqrt{R}\begin{pmatrix} e^{-\frac{i}{2}(\alpha+\gamma)} \cos\frac{\beta}{2} \\ e^{\frac{i}{2}(\alpha-\gamma)} \sin\frac{\beta}{2} \end{pmatrix}, \tag{4.248}$$

and

$$\omega = \left(i\vec{P}\cdot\vec{\tau} + \frac{q}{R}\tau_0\right)\pi. \tag{4.249}$$

In order to compute the symplectic structure and moment maps in terms of $\vec{R}$ and $\vec{P}$, we note that

$$\pi^\dagger\pi = R, \quad \pi^\dagger\vec{\tau}\pi = \vec{R}, \tag{4.250}$$

where

$$\vec{R} = (X_1, X_2, X_3) = (R\sin\beta\cos\alpha, R\sin\beta\sin\alpha, R\cos\beta), \tag{4.251}$$

and

$$\omega^\dagger\omega = R\vec{P}^2 + \frac{q}{R}, \quad \omega^\dagger\vec{\tau}\omega = -2\vec{P}\times\vec{J} + \left(R\vec{P}^2 + \frac{q}{R}\right)\hat{R}. \tag{4.252}$$

It is not difficult to check that, for fixed q, the twistor $Z = \binom{\omega}{\pi}$ satisfies (4.246) and thus belongs to $\mathbb{T}_q$. Moreover, the diagonal $U(1)$ acts simply by shifting γ, so that the vectors $\vec{P}, \vec{R} \in \mathbb{R}^3$, which are independent of γ, are good coordinates on the quotient $\tilde{\mathcal{M}}_q$.

The symplectic structure (4.239) induces a symplectic structure on $\tilde{\mathcal{M}}_q$ which can be expressed as

$$\Omega = dX_l \wedge dP_l + \frac{q}{2R^3}\epsilon_{iln}X_n dX_i \wedge dX_l. \tag{4.253}$$

The moment maps for γ_{50}, γ_{60} and the generators of their stabiliser Lie algebras can be written in terms of $\vec{P}, \vec{R}$ as

$$\begin{aligned}
Z^*\gamma_{50}Z &= \frac{1}{2}(\omega^\dagger\omega - \pi^\dagger\pi) = \frac{1}{2}\left(R\vec{P}^2 + \frac{q}{R} - R\right), \\
Z^*\gamma_{06}Z &= \frac{1}{2}(\pi^\dagger\pi + \omega^\dagger\omega) = \frac{1}{2}\left(R\vec{P}^2 + \frac{q}{R} + R\right), \\
Z^*\vec{s}Z &= \frac{1}{2}(\omega^\dagger\vec{\tau}\pi + \pi^\dagger\vec{\tau}\omega) = \vec{R}\times\vec{P} + q\hat{R}, \\
Z^*\vec{t}Z &= -\frac{1}{2}(\pi^\dagger\vec{\tau}\pi + \omega^\dagger\vec{\tau}\omega) = \vec{P}\times\vec{J} - \frac{1}{2}\left(R\vec{P}^2 + \frac{q}{R} + R\right)\hat{R}, \\
Z^*\vec{r}Z &= \frac{1}{2}(\pi^\dagger\vec{\tau}\pi - \omega^\dagger\vec{\tau}\omega) = \vec{P}\times\vec{J} - \frac{1}{2}\left(R\vec{P}^2 + \frac{q}{R} - R\right)\hat{R}.
\end{aligned} \tag{4.254}$$

We can summarise these formulae more neatly by introducing a variable κ which can take the values ± 1. Then we write

$$\tilde{H}_p = \frac{1}{2}\left(R\vec{P}^2 + \frac{q}{R} + \kappa R\right), \tag{4.255}$$

for the moment maps $Z^*\gamma_{50}Z$ and $Z^*\gamma_{06}Z$ with $\kappa = 1$ and $\kappa = -1$. We write

$$\vec{J} = \vec{R}\times\vec{P} + q\hat{R}, \tag{4.256}$$

for the vector or moment map $Z^*\vec{s}Z$, and finally note that

$$\vec{K} = \vec{P}\times\vec{J} - \tilde{H}_p\hat{R} \tag{4.257}$$

unifies the moment maps $Z^*\vec{t}Z$ and $Z^*\vec{r}Z$ for $\kappa = 1$ and $\kappa = -1$.

It then follows from the general theory of moment maps (and can also be verified directly) that the Poisson brackets of these moment maps are, up to factors of i, the commutators of the Lie algebra elements which enter the definition. In other words, the brackets are

$$\{\tilde{H}_p, J_i\} = \{\tilde{H}_p, K_i\} = 0$$
$$\{J_i, J_j\} = \epsilon_{ijk} J_k, \qquad \{J_i, K_j\} = \epsilon_{ijk} K_k, \qquad \{K_i, K_j\} = \kappa \epsilon_{ijk} J_k. \tag{4.258}$$

Adapting the treatment of [46], we shall show how the quotient $\tilde{\mathcal{M}}_q$ (4.247) can be mapped onto the symplectic quotient $\mathcal{M}_q$ (4.106) of the cotangent bundle of TN. The map between the phase spaces is not canonical, but it can be extended to a map on the evolution space, preserving the presymplectic (or Poincaré-Cartan) 2-form.

For a phase space $\mathcal{M}$ with symplectic structure ω and Hamiltonian H, the evolution space is $\mathcal{M} \times \mathbb{R}$, and the presymplectic 2-form is $\omega + dH \wedge dt$, where t is a global (time) coordinate on $\mathbb{R}$. The trajectories of the flow with Hamiltonian H can be characterised as the vortex lines of $\omega + dH \wedge dt$, i.e., the lines whose tangent lines are in the null space of $\omega + dH \wedge dt$. Now consider extended phase spaces $\mathcal{M} \times \mathbb{R}$ and $\tilde{\mathcal{M}} \times \tilde{\mathbb{R}}$ with symplectic structures and Hamiltonians (ω, H) on $\mathcal{M}$ and $(\tilde{\omega}, \tilde{H})$ on $\tilde{\mathcal{M}}$, and time coordinates t on $\mathbb{R}$ and $\tilde{t}$ on $\tilde{\mathbb{R}}$. Then a map

$$F : \mathcal{M} \times \mathbb{R} \to \tilde{\mathcal{M}} \times \tilde{\mathbb{R}} \tag{4.259}$$

which satisfies

$$F^*(\tilde{\omega} + d\tilde{H} \wedge d\tilde{t}) = \omega + dH \wedge dt \tag{4.260}$$

will map trajectories in the Hamiltonian system $(\mathcal{M}, \omega, H)$ to trajectories in the Hamiltonian system $(\tilde{\mathcal{M}}, \tilde{\omega}, \tilde{H})$. For details and a pedagogical discussion of Hamiltonian trajectories as vortex lines of Poincaré-Cartan structures see [47].

We should stress that, in contrast to the treatment of the ungauged case with negative L in [46], and unlike in the usual Kepler problem, no regularisation is required in our case since our Hamiltonian is smooth and finite on the entire phase space. For definiteness we focus on the case

$$L^2 H_p < \left(q + \frac{p}{2}\right)^2, \tag{4.261}$$

which is relevant for bounded orbits.

The two extended phase spaces we would like to map into each other are $\mathcal{M}_q \times \mathbb{R}$ with presymplectic 2-form

$$\sigma = \omega + dH_p \wedge dt = dx_l \wedge dp_l + \frac{q}{2r^3}\epsilon_{iln}x_n dx_i \wedge dx_l + dH_p \wedge dt, \tag{4.262}$$

and $\tilde{\mathcal{M}}_q \times \tilde{\mathbb{R}}$ with presymplectic 2-form

$$\Sigma = dX_l \wedge dP_l + \frac{q}{2R^3}\epsilon_{iln}X_n dX_i \wedge dX_l + d\tilde{H}_p \wedge d\tilde{t}. \tag{4.263}$$

The required map is most easily written down in terms of the coordinates $(\vec{P}, \vec{R}, \tilde{t})$ of $\tilde{\mathcal{M}}_q \times \tilde{\mathbb{R}}$ and the coordinates $(\vec{p}, \vec{r}, t)$ on $\mathcal{M}_q \times \mathbb{R}$. It takes the form

$$F : \mathcal{M}_q \times \mathbb{R} \to \tilde{\mathcal{M}}_q \times \tilde{\mathbb{R}}, \qquad (\vec{r}, \vec{p}, t) \mapsto (\vec{R}, \vec{P}, \tilde{t}), \tag{4.264}$$

where

$$\begin{aligned}
\vec{R} &= \vec{r}\sqrt{\frac{1}{L^2}\left(q + \frac{p}{2}\right)^2 - H_p}, \\
\vec{P} &= \frac{\vec{p}}{\sqrt{\frac{1}{L^2}\left(q + \frac{p}{2}\right)^2 - H_p}}, \\
\tilde{t} &= \frac{\sqrt{\frac{1}{L^2}\left(q + \frac{p}{2}\right)^2 - H_p}}{\frac{p^2}{L} + \frac{pq}{2L} - \frac{1}{2}LH_p}\left\{\vec{r}\cdot\vec{p} + 2\left[\frac{1}{L^2}\left(q + \frac{p}{2}\right)^2 - H_p\right]t\right\}.
\end{aligned} \tag{4.265}$$

A lengthy calculation shows

$$\begin{aligned}
F^*(dX_l \wedge dP_l) &= dx_l \wedge dp_l + \frac{\frac{1}{2}(\vec{p}\cdot d\vec{r} + \vec{r}\cdot d\vec{p}) \wedge dH_p}{\frac{1}{L^2}(q + \frac{p}{2})^2 - H_p}, \\
F^*\left(\frac{q}{2R^3}\epsilon_{iln}X_n dX_i \wedge dX_l\right) &= \frac{q}{2r^3}\epsilon_{iln}x_n dx_i \wedge dx_l,
\end{aligned} \tag{4.266}$$

and

$$F^*(d\tilde{H}_p \wedge d\tilde{t}) = dH_p \wedge dt - \frac{\frac{1}{2}(\vec{p}\cdot d\vec{r} + \vec{r}\cdot d\vec{p}) \wedge dH_p}{\frac{1}{L^2}\left(q + \frac{p}{2}\right)^2 - H_p}. \tag{4.267}$$

Combining these, we deduce

$$F^*\Sigma = \sigma, \tag{4.268}$$

as claimed. It follows that F maps solutions of the Hamilton equations

$$\frac{d\vec{r}}{dt} = \frac{\partial H_p}{\partial \vec{p}}, \qquad \frac{d\vec{p}}{dt} = -\frac{\partial H_p}{\partial \vec{r}} \tag{4.269}$$

to solutions of the Hamilton equations

$$\frac{d\vec{R}}{d\tilde{t}} = \frac{\partial \tilde{H}_p}{\partial \vec{P}}, \qquad \frac{d\vec{P}}{d\tilde{t}} = -\frac{\partial \tilde{H}_p}{\partial \vec{R}}. \tag{4.270}$$

Having seen that F maps trajectories to trajectories, albeit traversed at different rates, we conclude this section by showing how F relates observables. It is easy to check that F maps the angular momentum in $\tilde{\mathcal{M}}_q$ to the the angular momentum in $\mathcal{M}_q$, i.e., the substitution of (4.265) into (4.256) gives the TN angular momentum

$$\vec{J} = \vec{r} \times \vec{p} + q\hat{r}. \tag{4.271}$$

The Hamiltonians and the Runge-Lenz generators, however, are related by pulling back with F together with rescaling. Substituting the expressions (4.265) into the Hamiltonian (4.255) with $\kappa = 1$, one finds the re-scaled gauged TN Hamiltonian,

$$\tilde{H}_p = \frac{L^2 H_p - 2q^2 - pq}{2L\sqrt{\frac{1}{L^2}(q + \frac{p}{2})^2 - H_p}}, \tag{4.272}$$

and the substitution into (4.257) (again with $\kappa = 1$) gives a rescaled Runge-Lenz vector (4.214),

$$\vec{K} = \frac{\vec{p} \times \vec{J} - \frac{1}{2L}(L^2 H_p - 2q^2 - pq)\hat{r}}{\sqrt{\frac{1}{L^2}(q + \frac{p}{2})^2 - H_p}}. \tag{4.273}$$

Chapter 5

Euclidean Schwarzschild Zero-Modes

5.1 Euclidean Schwarzschild Geometry

5.1.1 Vacuum Einstein equations

The Schwarzschild solution of the vacuum Einstein field equations describes the gravitational field of an spherical symmetric body. This space describes a black hole that has no angular momentum or electric charge and is characterised by its mass only. Hawking [48] derived a Euclidean version of this space to argue for the thermal nature of particle creation at a Schwarzschild black hole. The Euclideanized version is obtained by taking the time coordinate to be imaginary $t = i\tau$ and so, in the standard spherical coordinates, the metric takes the form

$$ds^2 = c^2 d\tau^2 + f^2 dr^2 + a^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{5.1}$$

where the quantities a, c and f are radial functions given by the condition that the Ricci tensor vanishes. This manifold has the non-trivial topology $\mathbb{R}^2 \times S^2$ [49].

In order to gain a better understanding of the geometry of ES and to derive useful quantities for the computation of the Dirac operator on this space, we compute the curvature and Ricci tensors and use them to derive the functions a, b, c. In the following we adapt the standard Lorentzian treatment [50].

First we observe that the metric admits the vierbein

$$e^1 = a d\theta, \quad e^2 = a \sin\theta d\phi, \quad e^3 = f dr, \quad e^4 = c d\tau, \tag{5.2}$$

and that the dual vector fields satisfying the relation $(e^i, E_j) = \delta^i_j$ are

$$E_1 = \frac{1}{a}\partial_\theta, \quad E_2 = \frac{1}{a\sin\theta}\partial_\phi, \quad E_3 = \frac{1}{f}\partial_r, \quad E_4 = \frac{1}{c}\partial_\tau. \tag{5.3}$$

The next step is to compute a connection on this space. We will work with the spin connection since the calculation of the curvature tensor is relatively easy in this case. To do this we compute to obtain

$$\begin{aligned} de^1 &= \frac{a'}{af} e^3 \wedge e^1, \\ de^2 &= \frac{a'}{af} e^3 \wedge e^2 + \frac{\cos\theta}{a\sin\theta} e^1 \wedge e^2, \\ de^3 &= 0, \\ de^4 &= \frac{c'}{cf} e^3 \wedge e^4, \end{aligned} \tag{5.4}$$

where the prime denotes radial derivation. Then using the defining equations of the spin connection (2.12) we find the non-vanishing components

$$\omega^3{}_1 = -\frac{a'}{af} e^1, \quad \omega^3{}_2 = -\frac{a'}{af} e^2, \quad \omega^1{}_2 = -\frac{\cos\theta}{a\sin\theta} e^2, \quad \omega^4{}_3 = \frac{c'}{cf} e^4. \tag{5.5}$$

With these we compute the components of the curvature $R^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$,

$$\begin{aligned} R^4{}_3 &= -\frac{1}{ac}\frac{d}{dr}\left(\frac{c'}{f}\right) e^4 \wedge e^3, \quad R^4{}_1 = -\frac{a'c'}{acf^2} e^4 \wedge e^1, \\ R^4{}_2 &= -\frac{a'c'}{acf^2} e^4 \wedge e^2, \quad R^3{}_1 = -\frac{1}{af}\frac{d}{dr}\left(\frac{a'}{f}\right) e^3 \wedge e^1, \\ R^3{}_2 &= -\frac{1}{af}\frac{d}{dr}\left(\frac{a'}{f}\right) e^3 \wedge e^2, \quad R^1{}_2 = \frac{1}{a^2}\left[1 - \left(\frac{a'}{f}\right)^2\right] e^1 \wedge e^2. \end{aligned} \tag{5.6}$$

The components $R^i{}_{jkl} = R^i{}_j(E_k, E_l)$ are related to the curvature $R^\alpha{}_{\beta\mu\nu}$ of the Levi-Civita connection (2.14) as in (2.15). Using this relation, we see that the Ricci tensor $R_{\beta\nu} = R^{\alpha}{}_{\beta\alpha\nu}$ can be written as

$$\begin{aligned} R_{\beta\nu} &= E_i^{\alpha} e_{\alpha}^k e_{\beta}^j e_{\nu}^l R^i{}_{jkl} \\ &= e_{\beta}^j e_{\nu}^l R^k{}_{jkl}. \end{aligned} \tag{5.7}$$

Because the matrix $[e_{\alpha}^i]$ is diagonal, the Einstein vacuum equations $R_{\beta\mu} = 0$ are equivalent to $R_{jl} = R^k{}_{jkl} = 0$, and so considering the non-vanishing components

$$\begin{aligned} R^3{}_{434} &= R^4{}_{343} = -\frac{1}{ac}\frac{d}{dr}\left(\frac{c'}{f}\right), \\ R^1{}_{414} &= R^4{}_{141} = -\frac{a'c'}{acf^2}, \\ R^2{}_{424} &= R^4{}_{242} = -\frac{a'c'}{acf^2}, \\ R^1{}_{313} &= R^3{}_{131} = -\frac{1}{af}\frac{d}{dr}\left(\frac{a'}{f}\right), \\ R^2{}_{323} &= R^3{}_{232} = -\frac{1}{af}\frac{d}{dr}\left(\frac{a'}{f}\right), \\ R^2{}_{121} &= R^1{}_{212} = \frac{1}{a^2}\left[1 - \left(\frac{a'}{f}\right)^2\right], \end{aligned} \tag{5.8}$$

we see that there are up to 4 equations given by $R^k{}_{1k1} = R^k{}_{2k2} = R^k{}_{3k3} = R^k{}_{4k4} = 0$. Using (5.8) and noting that $R^k{}_{1k1} = R^k{}_{2k2}$ we obtain the three independent equations

$$\begin{aligned} \frac{1}{c}\frac{d}{dr}\left(\frac{c'}{f}\right) + \frac{2a'c'}{cf^2} &= 0, \\ \frac{1}{c}\frac{d}{dr}\left(\frac{c'}{f}\right) + \frac{2}{f}\frac{d}{dr}\left(\frac{a'}{f}\right) &= 0, \\ \frac{1}{f}\frac{d}{dr}\left(\frac{a'}{f}\right) + \frac{a'c'}{cf^2} - \frac{1}{a}\left[1 - \left(\frac{a'}{f}\right)^2\right] &= 0. \end{aligned} \tag{5.9}$$

From the first two equations we deduce

$$\frac{d}{dr}\left(\frac{a'}{f}\right) = \frac{a'c'}{cf}, \tag{5.10}$$

which implies

$$\frac{a'}{f} = kc, \tag{5.11}$$

with k a constant. The insertion of this into the third equation yields

$$\frac{c'}{f} = \frac{1}{2ak}(1 - c^2). \tag{5.12}$$

As a check we consider the case $a = r$ (which corresponds to the usual Euclidean Schwarzschild metric). In this case equation (5.11) becomes $cf = k^{-1}$. Rescaling the time so that $k = 1$ then f and c become each others inverse $cf = 1$. Using this in (5.12) we obtain

$$\frac{d}{dr}c^2 - \frac{1}{r}(1 - c^2) = 0, \tag{5.13}$$

which can be easily integrated to obtain $c^2 = 1 - \frac{L}{r}$ with L a constant. We therefore deduce the solutions

$$c = \sqrt{1 - \frac{L}{r}}, \quad f = \frac{1}{\sqrt{1 - \frac{L}{r}}} \tag{5.14}$$

which give the familiar Euclidean Schwarzschild metric

$$ds^2 = V d\tau^2 + \frac{1}{V}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad V = 1 - \frac{L}{r}. \tag{5.15}$$

The parameter τ is chosen to be periodic [48]. In order to work out the period we consider the first two terms of the metric

$$ds^2 = V d\tau^2 + \frac{1}{V}dr^2, \tag{5.16}$$

in the vicinity of $r = L$. It will be convenient then to write r in terms of a new parameter $\rho \ll 1$ as $r = L + \rho$. Then using the approximation $V \cong \frac{\rho}{L}$ we have

$$ds^2 = \frac{L}{\rho}d\rho^2 + \frac{\rho}{L}d\tau^2. \tag{5.17}$$

We can simplify this further by introducing a radial coordinate R given by $L\rho = \frac{1}{4}R^2$ which implies

$$L d\rho = \frac{1}{2}R dR. \tag{5.18}$$

Using this we obtain the $U(1)$ invariant metric

$$ds^2 = \frac{4L^2}{R^2}\frac{R^2}{4L^2}dR^2 + \frac{R^2}{4L^2}d\tau^2 = dR^2 + R^2\left(\frac{d\tau}{2L}\right)^2. \tag{5.19}$$

Observe that this 2-manifold is like the cigar-shaped surface of the toy model (4.67). The proper radial distance R can be defined in general as

$$R = \int_L^r \frac{1}{\sqrt{V}} dr'. \tag{5.20}$$

In order for $\frac{\tau}{2L}$ to lie in the range $0 \leq \frac{\tau}{2L} \leq 2\pi$ we require

$$0 \leq \tau \leq 4L\pi, \tag{5.21}$$

which shows that the period of τ is $4\pi L$.

The region

$$L \leq r, \quad 0 \leq \tau \leq 4\pi L, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi, \tag{5.22}$$

defines a manifold which is regular everywhere [51]. The 2-sphere $r = L$ is called a bolt.

5.1.2 Instantons on Euclidean Schwarzschild

In Chapter 3 we saw that TN admits an harmonic form which is the Poincaré dual to the 2-sphere and may be interpreted as the field of the electron. This 2-form also plays an important role in the model of the spin of the electron given the Dirac operator as it only has a non trivial kernel when coupled to its gauge potential (3.106). ES also admits a self-dual, square integrable and rotationally invariant 2-form [7, 49]. It is not exact, but locally given by the exterior derivative of the $U(1)$ potential

$$A = -i\frac{p}{2}\left(\frac{1}{r}d\tau - \cos\theta d\phi\right), \tag{5.23}$$

which is singular in both the north and south poles. Thus the 2-form

$$\begin{aligned} F = dA &= \frac{ip}{2}\left(\frac{1}{r^2}dr \wedge d\tau - \sin\theta d\theta \wedge d\phi\right), \\ &= \frac{ip}{2r^2}(e^3 \wedge e^4 - e^1 \wedge e^2), \end{aligned} \tag{5.24}$$

is not globally exact. A plays a similar role to the potential (3.106) of the TN case in the sense that the Dirac operator on ES only has a non trivial kernel when coupled to it. As we shall see, this is due to the fact that the Dirac operator on ES contains the Dirac operator on the 2-sphere, which when coupled to the restriction $A|_{S^2}$ (see Sect. 3.1.4) has a non trivial kernel given by irreducible $SU(2)$ representations of dimension $|p|$.

The singularities of the potential (5.23) can be removed by adding the exact form $d(\mp\frac{ip}{2}\phi)$ to it. Thus the upper and lower signs give two potentials

$$A_N = -i\frac{p}{2}\left[\frac{1}{r}d\tau + (1-\cos\theta)d\phi\right], \quad A_S = -i\frac{p}{2}\left[\frac{1}{r}d\tau - (1+\cos\theta)d\phi\right], \tag{5.25}$$

which are well defined on the northern and southern hemispheres respectively. They are related to each other by the $U(1)$ gauge transformation

$$A_N = A_S + \gamma^{-1}d\gamma, \tag{5.26}$$

where $\gamma = e^{-ip\phi}$ and hence p is an integer by the Dirac quantisation condition.

5.2 Zero modes on Euclidean Schwarzschild

5.2.1 Twisted Dirac operator on Euclidean Schwarzschild

In this section we compute the zero-modes of the Dirac operator on ES coupled to the gauge potential (5.53) by using both spherical and complex coordinates. We confirm that the space of normalised zero-modes has dimension $|p|^2$ in agreement with [7]. Assuming $p > 0$ we find that in the case $p = 2$ one gets two doublets of spin-$\frac{1}{2}$ states. For general p we get $|p|$ dimensional irreducible representations of $SU(2)$ of multiplicity $|p|$.

In order to keep things general we compute the Dirac operator associated to the metric (5.1) and then substitute $a = r$ and the solutions (5.14) which corresponds to the usual Euclidean Schwarzschild metric. We are going to use the definition (2.17) along with the following representation for the γ-matrices:

$$\gamma^i = \begin{pmatrix} 0 & \tau_i \\ -\tau_i & 0 \end{pmatrix}, \gamma^4 = \begin{pmatrix} 0 & -i\tau_0 \\ -i\tau_0 & 0 \end{pmatrix}, \tag{5.27}$$

which satisfy the relations

$$[\gamma^i, \gamma^j] = -2i\epsilon_{ijk} \begin{pmatrix} \tau_k & 0 \\ 0 & \tau_k \end{pmatrix}, \quad [\gamma^4, \gamma^i] = 2i \begin{pmatrix} \tau_i & 0 \\ 0 & -\tau_i \end{pmatrix}. \tag{5.28}$$

Using the components of the spin connection (5.5), we compute the components of Γ according to (2.9),

$$\Gamma_1 = \begin{pmatrix} -\frac{ia'}{2f}\tau_2 & 0 \\ 0 & -\frac{ia'}{2f}\tau_2 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} -\frac{i\cos\theta}{2}\tau_3 + \frac{ia'\sin\theta}{2f}\tau_1 & 0 \\ 0 & -\frac{i\cos\theta}{2}\tau_3 + \frac{ia'\sin\theta}{2f}\tau_1 \end{pmatrix},$$
$$\Gamma_4 = \begin{pmatrix} -\frac{ic'}{2f}\tau_3 & 0 \\ 0 & \frac{ic'}{2f}\tau_3 \end{pmatrix}. \tag{5.29}$$

With these ingredients we compute the Dirac operator

$$\not{D}_{ES} = \begin{pmatrix} 0 & T^\dagger \\ T & 0 \end{pmatrix}, \tag{5.30}$$

in which

$$\begin{aligned} T &= -\tau_i\partial_i - i\tau_0\partial_4 - \frac{\cos\theta}{2a\sin\theta}\tau_1 - \left(\frac{a'}{af} + \frac{c'}{2cf}\right)\tau_3, \\ T^\dagger &= \tau_i\partial_i - i\tau_0\partial_4 + \frac{\cos\theta}{2a\sin\theta}\tau_1 + \left(\frac{a'}{af} + \frac{c'}{2cf}\right)\tau_3. \end{aligned} \tag{5.31}$$

Coupling the operator to the potential

$$A = -i\frac{p}{2}\left(\frac{1}{a}d\tau - \cos\theta d\phi\right), \tag{5.32}$$

which reduces to (5.23) in the case $a = r$, gives

$$T_p = -\begin{pmatrix} \frac{1}{f}\partial_r + \frac{i}{c}\partial_\tau + \frac{p}{2ac} + \frac{a'}{af} + \frac{c'}{2cf} & \frac{1}{a}[\partial_\theta - i\csc\theta\partial_\phi + (\frac{p+1}{2})\cot\theta] \\ \frac{1}{a}[\partial_\theta + i\csc\theta\partial_\phi - (\frac{p-1}{2})\cot\theta] & -\frac{1}{f}\partial_r + \frac{i}{c}\partial_\tau + \frac{p}{2ac} - \frac{a'}{af} - \frac{c'}{2cf} \end{pmatrix}, \tag{5.33}$$

$$T_p^\dagger = \begin{pmatrix} \frac{1}{f}\partial_r - \frac{i}{c}\partial_\tau - \frac{p}{2ac} + \frac{a'}{af} + \frac{c'}{2cf} & \frac{1}{a}[\partial_\theta - i\csc\theta\partial_\phi + (\frac{p+1}{2})\cot\theta] \\ \frac{1}{a}[\partial_\theta + i\csc\theta\partial_\phi - (\frac{p-1}{2})\cot\theta] & -\frac{1}{f}\partial_r - \frac{i}{c}\partial_\tau - \frac{p}{2ac} - \frac{a'}{af} - \frac{c'}{2cf} \end{pmatrix}. \quad (5.34)$$

Here we identify the edth operators (3.28) in spherical coordinates:

$$\begin{aligned} \eth_s &= \partial_\theta + i\csc\theta\partial_\phi - s\cot\theta, \quad s = \frac{p-1}{2}, \\ \bar{\eth}_{\tilde{s}} &= \partial_\theta - i\csc\theta\partial_\phi + \tilde{s}\cot\theta \quad \tilde{s} = \frac{p+1}{2}. \end{aligned} \quad (5.35)$$

When acting on the spin-weighted spherical harmonics Y^j_{sm} [20] they satisfy the following relations:

$$\begin{aligned} \eth_s Y^j_{sm} &= [(j-s)(j+s+1)]^{1/2} Y^j_{s+1,m}, \\ \bar{\eth}_{\tilde{s}} Y^j_{\tilde{s}m} &= -[(j+\tilde{s})(j-\tilde{s}+1)]^{1/2} Y^j_{\tilde{s}-1,m}, \end{aligned} \quad (5.36)$$

which are analogous of (3.22), (3.24). Using these and noting that $\tilde{s} = s+1$ it follows that

$$\begin{aligned} \bar{\eth}_{\tilde{s}}\eth_s Y^j_{s,m} &= [-j(j+1) + s(s+1)]Y^j_{s,m} = \kappa Y^j_{s,m}, \\ \eth_s\bar{\eth}_{\tilde{s}} Y^j_{\tilde{s}m} &= [-j(j+1) + \tilde{s}(\tilde{s}-1)]Y^j_{\tilde{s}m} = \kappa Y^j_{\tilde{s}m}, \end{aligned} \quad (5.37)$$

where $\kappa = -j(j+1) + \frac{p^2}{4} - \frac{1}{4}$. These relations will be useful in our discussion of the Laplace operator on ES latter on.

5.2.2 Zero-modes

In order to compute zero-modes of the Dirac operator

$$\not{D}_{ES,p}\Psi = 0, \quad (5.38)$$

we assume solutions of the form,

$$\Psi = \begin{pmatrix} R_1(r)e^{i\omega\tau}Y^j_{sm} \\ R_2(r)e^{-i\omega\tau}Y^j_{\tilde{s}\tilde{m}} \\ R_3(r)e^{i\omega\tau}Y^j_{sm} \\ R_4(r)e^{-i\omega\tau}Y^j_{\tilde{s}\tilde{m}} \end{pmatrix}. \tag{5.39}$$

Spinorial fields in 4D should satisfy [7] the property $\Psi(\tau + 4L\pi) = -\Psi(\tau)$ which implies that ω is quantised:

$$\omega = \frac{2n+1}{4L}, \quad n \text{ an integer.} \tag{5.40}$$

For Ψ to be a zero-mode, the simplest possibility is for Y^j_{sm} to lie in the kernel of $\eth_s$ and so we see from (5.36) that this is true if $j = s = (p-1)/2$. Since j has to be positive this is only possible for $p \geq 1$. Similarly the function $Y^j_{\tilde{s}\tilde{m}}$ has to lie in the kernel of $\bar{\eth}$ and in this case $j = -\tilde{s} = -(p+1)/2$. The condition $j \geq 0$ now requires $p \leq -1$. So for example in the case $p \geq 1$, only the first and third components of Ψ are not trivial, and then inserting Ψ into (5.38) leads to the radial equation

$$\frac{1}{f}\partial_r R_1 + \left(-\frac{\omega}{c} + \frac{p}{2ac} + \frac{a'}{af} + \frac{c'}{2cf}\right) R_1 = 0, \tag{5.41}$$

with a similar equation for R_3 with the signs of p and ω reversed. Observe that the preceding equation can be recast as

$$\int \frac{dR_1}{R_1} = \int \left[\frac{\omega f}{c} - \frac{pf}{2ac} - \frac{d}{dr}\left(\ln a + \frac{1}{2}\ln c\right)\right] dr, \tag{5.42}$$

which can be integrated to obtain

$$R_1(r) = \frac{\alpha_1}{a\sqrt{c}} e^{\int\left(\frac{\omega f}{c} - \frac{pf}{2ac}\right)dr}, \tag{5.43}$$

where α_1 is a constant. Using this we deduce the general form of the zero-modes for p positive

$$\Psi = \begin{pmatrix} \frac{\alpha_1}{a\sqrt{c}} e^{i\omega\tau} e^{\int\left(\frac{\omega f}{c} - \frac{pf}{2ac}\right)dr} e^{i\omega\tau} Y^j_{jm} \\ 0 \\ \frac{\alpha_3}{a\sqrt{c}} e^{i\omega\tau} e^{\int\left(-\frac{\omega f}{c} + \frac{pf}{2ac}\right)dr} e^{i\omega\tau} Y^j_{jm} \\ 0 \end{pmatrix}, \quad p \geq 1. \tag{5.44}$$

The substitution of $a = r$ and the solutions (5.14) shows that the radial contribution of the top component is $e^{r\omega} r^{-\frac{3}{4}} (r-L)^{\left(-\frac{1}{4} - \frac{p}{2} + L\omega\right)}$. From this we can see that in order to have square integrability at infinity we need $\omega \leq 0$, but in this makes the radial function not square integrable near $r = L$. So we are left with a solution of the form

$$\Psi = \begin{pmatrix} 0 \\ 0 \\ \alpha_3 e^{i\omega\tau} e^{-r\omega} r^{-\frac{3}{4}} (r-L)^{\left(-\frac{1}{4} + \frac{p}{2} - L\omega\right)} Y^j_{jm} \\ 0 \end{pmatrix}, \quad p \geq 1, \quad j = \frac{p-1}{2}. \tag{5.45}$$

Now the square integrability condition at infinity requires $\omega \geq 0$ which from (5.40) implies $n \geq 0$. The condition for square integrability near $r = L$ sets an upper boundary for n since in this case we need

$$-\frac{1}{4} + \frac{p}{2} - L\omega = -\frac{1}{4} + \frac{p}{2} - \frac{2n+1}{4} > -\frac{1}{2}, \tag{5.46}$$

which implies $n < p$. Because p is an integer this is equivalent to $n \leq p-1$ and so altogether $0 \leq n \leq p-1$, which shows that n can take p different values. On the other hand, the fact that $j = (p-1)/2$ shows that the magnetic number $-j \leq m \leq j$ can take p different values too and hence the dimension of the zero-modes is p^2.

The computation of the zero-modes in the case $p \leq -1$ works in exactly the same way and a similar analysis shows that

$$\Psi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \alpha_4 e^{-i\omega\tau} e^{-r\omega} r^{-\frac{3}{4}} (r-L)^{\left(-\frac{1}{4}-\frac{p}{2}-L\omega\right)} Y^j_{-jm} \end{pmatrix}, \quad p \leq -1, \quad -j = \frac{p+1}{2}, \quad (5.47)$$

are regular zero modes provided $0 \leq n \leq -p-1$, which implies that n can take $|p|$ different values. Also the fact that $j = -(p+1)/2$ shows that m can take $|p|$ different values too, and hence the dimension of the zero-modes is $|p|^2$ as before.

5.2.3 Zero-modes in complex coordinates

The fact that the spin-weighted spherical harmonics appear in the zero modes of the previous section is due to the fact that the off-diagonal components of (5.33) and (5.34) are essentially the Dirac operator on the 2-sphere coupled to the Dirac monopole. This operator was studied in Chapter 3 where, using spherical coordinates, it was shown that its zero-modes form an irreducible representation of $SU(2)$. We now implement complex coordinates and use this result to show that the zero-modes on ES have this property too.

We are going to work with the following complex coordinate

$$z = \tan\frac{\theta}{2} e^{i\phi}, \quad \bar{z} = \tan\frac{\theta}{2} e^{-i\phi}, \quad (5.48)$$

which is regular on the north pole of the 2-sphere and is the analogous to (2.65). The gauge potential (5.23) is not compatible with this coordinate, for it is singular in the north pole. However we can use A_N (5.25) instead, which is well defined on the northern hemisphere. Noting that

$$\cos\theta = \frac{1-z\bar{z}}{1+z\bar{z}}, \quad d\phi = \frac{1}{2iz\bar{z}}(\bar{z}dz - zd\bar{z}), \quad (5.49)$$

we see that the potential A_N can be recast as follows:

$$A_N = -\frac{ip}{2a}d\tau + \frac{p}{2q}(zd\bar{z} - \bar{z}dz), \tag{5.50}$$

where we need to keep in mind that $a = r$.

Using the same procedure as before we find that in terms of the coordinate (5.48) the components of the Dirac operator (5.30) coupled to the potential A_N read

$$T_p = -\begin{pmatrix} \frac{1}{f}\partial_r + \frac{i}{c}\partial_\tau + \frac{p}{2ac} + \frac{a'}{af} + \frac{c'}{2cf} & \frac{1}{a}[q\partial_z - \frac{1}{2}(p+1)\bar{z}] \\ \frac{1}{a}[q\bar{\partial}_z + \frac{1}{2}(p-1)z] & -\frac{1}{f}\partial_r + \frac{i}{c}\partial_\tau + \frac{p}{2ac} - \frac{a'}{af} - \frac{c'}{2cf} \end{pmatrix},$$
$$T_p^\dagger = \begin{pmatrix} \frac{1}{f}\partial_r - \frac{i}{c}\partial_\tau - \frac{p}{2ac} + \frac{a'}{af} + \frac{c'}{2cf} & \frac{1}{a}[q\partial_z - \frac{1}{2}(p+1)\bar{z}] \\ \frac{1}{a}[q\bar{\partial}_z + \frac{1}{2}(p-1)z] & -\frac{1}{f}\partial_r - \frac{i}{c}\partial_\tau - \frac{p}{2ac} - \frac{a'}{af} - \frac{c'}{2cf} \end{pmatrix}. \tag{5.51}$$

Here we identify the off-diagonal components with the operators $\partial_s^\uparrow, \partial_{\tilde{s}}^\downarrow$ (3.17),

$$\begin{aligned} \partial_s^\uparrow &= \bar{\partial}_z + sz, \quad s = \frac{1}{2}(p-1), \\ \partial_{\tilde{s}}^\downarrow &= q\partial_z - \tilde{s}\bar{z}, \quad \tilde{s} = \frac{1}{2}(p+1), \end{aligned} \tag{5.52}$$

which are related to the operators $\eth_s, \bar{\eth}_{\tilde{s}}$ (5.35) as shown in (3.29). From the discussion in Sect. 3.1.3, the zero-eigenvalues of $\partial_s^\uparrow, \partial_{\tilde{s}}^\downarrow$ are given by functions of the form $q^{\frac{-p+1}{2}}p_1(z)$ and $q^{\frac{p+1}{2}}p_2(\bar{z})$, where $q = 1 + z\bar{z}$. They are regular at $z = 0$ if $p_1(z)$ is a polynomial of degree $p-1$ which is only possible for $p \geq 1$ or if $p_2(\bar{z})$ is a polynomial of degree $-p-1$ in which case $p \leq -1$. It is also shown that they are local sections of the p-th power of the hyperplane bundle.

The computation of the radial part of the zero-modes works in the same way as before, and so looking for zero-modes of the form (5.39) with the spin harmonics replaced by the zero-eigenvalues of $\partial_s^\uparrow$ and $\partial_{\tilde{s}}^\downarrow$ we find the analogous of the solutions

(5.45) and (5.47),

$$\Psi = \begin{pmatrix} 0 \\ 0 \\ \alpha_3 e^{i\omega\tau} e^{-r\omega} r^{-\frac{3}{4}} (r-L)^{(-\frac{1}{4}+\frac{p}{2}-L\omega)} q^{\frac{-p+1}{2}} \sum_{k=0}^{p-1} a_k z^k \\ 0 \end{pmatrix}, \quad p \geq 1, \tag{5.53}$$

$$\Psi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \alpha_4 e^{-i\omega\tau} e^{-r\omega} r^{-\frac{3}{4}} (r-L)^{(-\frac{1}{4}-\frac{p}{2}-L\omega)} q^{\frac{p+1}{2}} \sum_{k=0}^{-p+1} \tilde{a}_k \bar{z}^k \end{pmatrix}, \quad p \leq -1, \tag{5.54}$$

respectively. From the previous analysis of the radial solutions, there are $|p|$ different values of ω that yield square integrable zero-modes. Using this, and the fact that the dimension of the complex polynomials is also $|p|$, we see that the zero-modes have dimension $|p|^2$ as expected. As shown in Sect. 3.1.4 these polynomials are irreducible representations of $SU(2)$ of spin $j = (p-1)/2$ for $p \geq 1$ and $j = (-p-1)/2$ for $p \leq 1$. Hence, the space of the above zero-modes splits into $|p|$ copies of $|p|$-dimensional representations of $SU(2)$.

Probability distribution

Focusing in the case $p \geq 1$ and using $k = m + j$, where $j = (p-1)/2$, the functional dependance of the zero-mode (5.53) can be written as

$$\begin{aligned} \Psi &= \alpha_3 e^{i\omega\tau} e^{-r\omega} r^{-\frac{3}{4}} (r-L)^{(-\frac{1}{4}+\frac{p}{2}-L\omega)} q^{-j} \sum_{m=-j}^{j} a_m z^{m+j} \\ &= \alpha_3 e^{i\omega\tau} e^{-r\omega} r^{-\frac{3}{4}} (r-L)^{(-\frac{1}{4}+\frac{p}{2}-L\omega)} \sum_{m=-j}^{j} \left(\cos\frac{\theta}{2}\right)^{j-m} \left(\sin\frac{\theta}{2}\right)^{j+m} e^{i(j+m)\phi}, \end{aligned} \tag{5.55}$$

where we have used (5.48). Picking a term of fixed m, we obtain the probability distribution

$$|\Psi|^2 \propto e^{-2r\omega} r^{-\frac{3}{2}-2j} (r-L)^{(-\frac{1}{2}+p-2L\omega)} (r-x_3)^{j+m} (r+x_3)^{j-m}, \tag{5.56}$$

where $(x_1, x_2, x_3) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$, which resembles the distribution of the zero-modes of TN (3.125).

Spin-$\frac{1}{2}$

Let us consider the case of spin-$\frac{1}{2}$ states i.e. $j = \frac{1}{2}$ which, assuming for simplicity $p \geq 0$, are obtained from (5.53) by picking $p = 2$. In this case n lies in the range $0 \leq n \leq 2-1$ and so we have $n = 0, 1$. From (5.40) the corresponding values of ω are $\omega = \frac{1}{4L}, \frac{3}{4L}$ consecutively. Inserting these values in the zero-mode (5.53) we obtain two spin-$\frac{1}{2}$ doublets

$$\Psi^1_{\frac{1}{2}} = e^{\frac{i\tau}{4L}} e^{-\frac{r}{4L}} r^{-\frac{3}{4}} (r-L)^{\frac{1}{2}} q^{-\frac{1}{2}} (a_0 + a_1 z), \quad \Psi^2_{\frac{1}{2}} = e^{\frac{i3\tau}{4L}} e^{-\frac{3r}{4L}} r^{-\frac{3}{4}} q^{-\frac{1}{2}} (a_0 + a_1 z). \tag{5.57}$$

Both states are exponentially localised at the bolt. At $r = L$ one of them is zero while the other has a finite value, as we can see from the plots of their r dependance in figure 5.1.

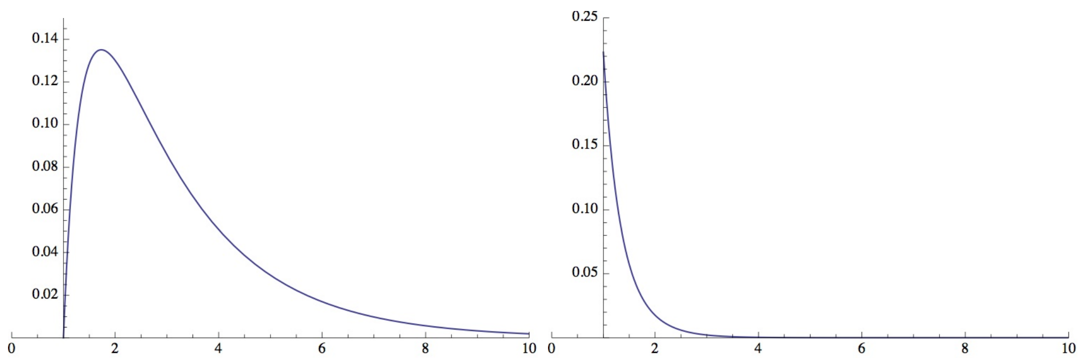

Figure 5.1: Plot of the r dependance of $|\Psi^1_{\frac{1}{2}}|^2$ (left) and $|\Psi^2_{\frac{1}{2}}|^2$ (right) with $L = 1$.

We notice from (5.1) and (5.15) that in ES the volume factor used to normalise the probability distribution is the same as in $\mathbb{R}^3$ i.e. $cfa^2 = r^2$. Thus the normalisation condition is

$$\int |\Psi(r)|^2 r^2 dr < \infty. \tag{5.58}$$

Thinking of $|\Psi|^2$ as a 'spin density', the above doublets give two admissible models for the spin degrees of freedom for the neutron - one of them with a spin density decaying faster than the other. This is reminiscent of the TN model for the electron, in which there is an extra spin state that cannot be eliminated by a logical condition. In the context of GMM, the extra spin-$\frac{1}{2}$ doublet can perhaps suggest that ES could be used as a dual model for the neutron and proton, with the degeneracy between the two doublets interpreted as isospin symmetry.

5.3 Euclidean Schwarzschild bound states

5.3.1 Laplace-Beltrami operator

Having discussed the zero-modes of the twisted Dirac operator, we now look at the other bound states of the Dirac Laplacian $\not{D}_p^\dagger \not{D}_p$. In the TN case, the computation of the non-zero-eigenvalues was possible due to the self-duality of the TN geometry, which implies the existence of a covariantly constant spinor. In the ES case, we do not have this simplification and for this reason we restrict our study to the case of the Laplace operator acting on scalar fields.

We can derive an action on a scalar field ϕ from the action of $\not{D}_p^\dagger \not{D}_p$ on $\phi\Psi$, where Ψ is a spinor. So using (2.26) and taking into account that in ES the scalar curvature R is zero, we see that

$$\not{D}_p^\dagger \not{D}_p \phi\Psi = -(g^{\mu\nu} D_\mu \partial_\nu \phi)\Psi - (g^{\mu\nu} D_\mu D_\nu \Psi)\phi + \frac{1}{2}[\gamma^i, \gamma^j] F_{ij} \phi\Psi. \tag{5.59}$$

It can be shown by using a direct calculation that in the first term the expression in brackets can be recast as

$$g^{\mu\nu} D_\mu D_\nu \phi = \Delta_p \phi, \tag{5.60}$$

where Δ_p is the gauged version of the Laplace-Beltrami operator [18],

$$\Delta = \frac{1}{\sqrt{g}} \partial_\mu g^{\mu\nu} \sqrt{g} \partial_\nu. \tag{5.61}$$

Also, using (5.24) and (5.27) it follows that the contribution of the curvature $F = dA$ is

$$\frac{1}{2}[\gamma^i, \gamma^j] F_{ij} = -\frac{p}{r^2} \begin{pmatrix} 0 & 0 \\ 0 & \tau_3 \end{pmatrix}. \tag{5.62}$$

From the above we observe that if there were a covariantly constant spinor $D_\nu \Psi =$

0 of the form $\Psi = \binom{\Psi_1}{0}$ then the eigenvalue problem $\not{D}_p^\dagger \not{D}_p \phi\Psi = \lambda\phi\Psi$ would reduce to solve the equation

$$-\Delta\phi = \lambda\phi. \tag{5.63}$$

However, we cannot simplify the problem in this way, for ES does not admit covariantly constant spinors. Nevertheless, we investigate the solutions of the previous equation as it is also an interesting problem. We leave the task of finding the solutions of the spinor eigenstates $\not{D}_p^\dagger \not{D}_p \Psi = \lambda\Psi$ for the future.

5.3.2 Bound states

In order to study the equation (5.63) in the ES space we observe that the Laplace-Beltrami operator (5.61) associated to the generalised ES metric (5.1) has the form

$$\Delta_{ES} = \frac{1}{a^2 cf}\partial_r \left(\frac{a^2 c}{f}\right)\partial_r + \frac{1}{c^2}\partial_\tau^2 + \frac{1}{a^2}\Delta_{S^2}, \tag{5.64}$$

in which Δ_{S^2} is the Laplace operator on the 2-sphere:

$$\Delta_{S^2} = \frac{1}{\sin^2\theta}\partial_\phi^2 + \frac{1}{\sin\theta}\partial_\theta \sin\theta\partial_\theta. \tag{5.65}$$

Minimally coupling the Laplace operator to the potential (5.32) yields

$$\Delta_{ES,p} = \frac{1}{a^2 cf}\partial_r \left(\frac{a^2 c}{f}\right)\partial_r + \frac{1}{c^2}\left(\partial_\tau - \frac{ip}{2a}\right)^2 + \frac{1}{a^2}\Delta_{S^2,p}, \tag{5.66}$$

where now

$$\Delta_{S^2,p} = \Delta_{S^2} + \frac{ip\cos\theta}{\sin^2\theta}\partial_\phi - \frac{p^2\cos^2\theta}{4\sin^2\theta}. \tag{5.67}$$

We can realise $\Delta_{S^2,p}$ in terms of the Edth operators $\eth, \bar{\eth}$. To determine the explicit relation we use (5.35) to compute

$$\bar{\eth}_{s+1}\eth_s = \Delta_{S^2} + \frac{2is\cos\theta}{\sin^2\theta}\partial_\phi - s^2\frac{\cos^2\theta}{\sin^2\theta} + s, \quad \eth_{s-1}\bar{\eth}_s = \Delta_{S^2} + \frac{2is\cos\theta}{\sin^2\theta}\partial_\phi - s^2\frac{\cos^2\theta}{\sin^2\theta} - s. \tag{5.68}$$

Then a comparison with $\Delta_{S^2,p}$ shows that, in the case $s = \frac{p}{2}$,

$$\Delta_{S^2,p} = \bar{\eth}_{\frac{p}{2}+1}\eth_{\frac{p}{2}} - \frac{p}{2} = \eth_{\frac{p}{2}-1}\bar{\eth}_{\frac{p}{2}} + \frac{p}{2}. \tag{5.69}$$

The ability to write the Laplace operator in this way allow us to compute the bound states of the the equation (5.63) $-\Delta_{ES,p}\phi = \lambda\phi$, by assuming solutions with the factorized form $\phi = \rho(r)e^{i\omega'\tau}Y^j_{s=\frac{p}{2}m}$. Using this we see that the requirement $-j \leq s \leq j$ implies the condition

$$j \geq \frac{|p|}{2}. \tag{5.70}$$

Notice that the value of j is not fixed as in the case of the zero-modes. Also the dependance on τ is not quantised as in (5.40) since now we have the condition $\phi(\tau + 4L\pi) = \phi(\tau)$, which implies

$$\omega' = \frac{n'}{2L}, \quad n' \text{ an integer.} \tag{5.71}$$

Using the ansatz for ϕ in the eigenvalue equation along with (5.14) and (5.37), we get the radial equation

$$V\frac{\partial^2\rho}{\partial r^2}+\left(\frac{L}{r^2}+\frac{2V}{r}\right)\frac{\partial\rho}{\partial r}-\frac{1}{V}\left(\omega'-\frac{p}{2r}\right)^2\rho+\frac{1}{r^2}\left[-j(j+1)+\frac{p^2}{4}\right]\rho+\lambda\rho = 0. \tag{5.72}$$

We have not been able to solve this equation exactly. However we can simplify the problem and solve it numerically by using the factorized solution

$$\rho = \frac{1}{a}\sqrt{\frac{f}{c}}g(r) = \frac{g(r)}{r\sqrt{V}}, \tag{5.73}$$

which gives a Schrödinger-like equation for the function $g(r)$:

$$Vg'' + (\lambda - \mathrm{v}(r))g = 0, \tag{5.74}$$

where

$$\mathrm{v}(r) = -\frac{L^2}{4r^4V} + \frac{1}{V}\left(\omega' - \frac{p}{2r}\right)^2 + \frac{1}{r^2}\left[j(j+1) - \frac{p^2}{4}\right]. \tag{5.75}$$

We can turn the above equation into the Sturm-Liouville form by dividing it by V,

$$-(Pg)'' + Qg = \lambda Wg, \tag{5.76}$$

in which $P = 1, Q = \mathrm{v}/V$ and $W = 1/V$. This equation can be solved numerically even when the function W is singular at $r = L$. There are several programs specialised in this kind of problems, one of them is SLEIGN2 [52], which is written in Fortran, and we use in the next section to compute the spectrum.

Asymptotic limits of $\boldsymbol{Q(r)}$

We now investigate the asymptotic form of the potential Q(r) at the endpoints of the interval (L, ∞). The asymptotic limits will be useful in the next section for the classification of the endpoints.

To obtain the form at the limit $r \to \infty$ we can use

$$\frac{1}{V} \simeq 1 + \frac{L}{r} + \frac{L^2}{r^2} + \frac{L^3}{r^3} + \frac{L^4}{r^4}. \tag{5.77}$$

Inserting this into $Q(r)$ and keeping terms up to order $O(r^{-2})$ we get

$$Q(r) \simeq \omega'^2 + \omega'(2L\omega' - p)\frac{1}{r} + (3L^2\omega'^2 - 2Lp\omega' + j(j+1))\frac{1}{r^2}. \tag{5.78}$$

We can see that in the case $p \geq 0$ the Coulomb term is attractive if

$$0 < L\omega' < \frac{p}{2}, \tag{5.79}$$

while for $p \leq 0$ we have,

$$\frac{p}{2} < L\omega' < 0. \tag{5.80}$$

To obtain the asymptotic form of the potential at $r \to L$ we introduce the coordinate $x = r - L$ which goes to zero at this limit. Doing that we see that for small x,

$$Q(x) \simeq \frac{1}{4L^2}\left[(2L\omega' - p)^2 - 1\right]\left(1 + \frac{L}{x}\right)^2. \tag{5.81}$$

The asymptotic limits (5.78) and (5.81) will be used in the next section to show that the endpoints of the interval (L, ∞), in which the problem (5.76) is defined, are of the limit-point kind. We also show that the eigenvalues can be computed by using the code SLEIGN2. It follows that solutions exist as long as the requirements

(5.79) and (5.80), which can be recast as

$$|L\omega'| < \left|\frac{p}{2}\right|, \tag{5.82}$$

are satisfied. This is analogous to the TN condition for bound states (4.175), which restrict the eigenvalues of the circle coordinate. As in TN, the bound states can be explained by the binding of the magnetic field (5.24) restricted to the cigar-shaped sub manifold (5.19) of ES.

We solve numerically the eigenvalue problem with the particular setting $L = 1, j = 5, p = 10, \omega' = 1$, which satisfy conditions (5.70) and (5.82). In this case the differential equation (5.76) reduces to

$$-g'' + \left(-\frac{8}{r} + \frac{13}{r^2}\right) g = (\lambda W - 1)g. \tag{5.83}$$

5.3.3 Computation of spectrum using SLEIGN2

In this section we use the SLEIGN2 code [52] to compute the spectrum of the Eigenvalue problem (5.76) in the interval (L, ∞). This program requires information on the endpoints i.e. whether they are limit-point, limit-circle, etc. We now show that both endpoints are limit-point by using the criteria of [53, 54], which says that an endpoint, for example L, is limit-point if for some eigenvalue $\lambda \in \mathbb{C}$ there exist at least one solution $g(r, \lambda)$ of (5.76) such that

$$\int_L^c W(r)|g(r, \lambda)|^2 dr = \infty, \tag{5.84}$$

where $L < c < \infty$, with a similar condition for the other endpoint. To do this, we evaluate the above integral with the solutions of the equation (5.76) that correspond to the asymptotic limits (5.78), (5.81) of the effective potential (5.75).

To analyse the endpoint $r = L$ we use the variable $x = r - L$ and observe that the biggest contribution to the potential (5.81) at $r = L$ (or equivalently $x = 0$) is $\frac{a}{x^2}$, where

$$a = \left(L\omega' - \frac{p}{2}\right)^2 - \frac{1}{4}. \tag{5.85}$$

Also at this limit $V \simeq x/L$ and hence $W \simeq L/x$. Using this in (5.74) we then get the equation

$$g'' - \frac{a}{x^2} g = 0. \tag{5.86}$$

Assuming solutions of the form $g = x^\beta$ we find that $\beta = \frac{1}{2} \pm (L\omega' - \frac{p}{2})$ and hence

$$g = A_\pm (r - L)^{\frac{1}{2} \pm (L\omega' - \frac{p}{2})}, \; A_\pm \text{ constants.} \tag{5.87}$$

Now using these solutions we compute

$$\begin{aligned} \int_L^c W(r)|g(r,\lambda)|^2 dr &= \int_0^{c-L} x^{\pm(2L\omega'-p)} dx \\ &\propto x^{\pm(2L\omega'-p)+1} \Big|_0^{c-L}. \end{aligned} \tag{5.88}$$

Notice that the integral diverges as long as the condition $|2L\omega' - p| > 1$ is satisfied. Since this is always possible then $r = L$ is a limit-point.

We now consider the second endpoint $r = \infty$ and notice from (5.78) that at this limit V and W converge to 1, and that the biggest contribution of the potential (5.78) is given by the the term ω'^2. Thus the eigenvalue equation (5.76) reduces to

$$g'' + (\lambda - \omega'^2) g = 0. \tag{5.89}$$

Assuming solutions of the form $g = e^{\alpha r}$ we obtain that $\alpha = \pm\sqrt{\omega'^2 - \lambda}$ and hence

$$g = A_\pm e^{\pm\sqrt{\omega'^2-\lambda} r}, \; A_\pm \text{ constants.} \tag{5.90}$$

Since $W = 1$ at this limit, the problem is equivalent to show that there is at least one solution which is not square integrable at infinity. This is true provided

$$\omega'^2 - \lambda > 0. \tag{5.91}$$

Having shown that the endpoints (L, ∞) of the problem (5.76) are of the limit-point kind we now implement this conditions and use SLEIGN2 to compute the first eigenvalues of the equation (5.83), which we show in table 5.1.

λ_0	: 0.313	λ_6	: 0.839
λ_1	: 0.500	λ_7	: 0.863
λ_2	: 0.622	λ_8	: 0.883
λ_3	: 0.705	λ_9	: 0.899
λ_4	: 0.763	λ_{10}	: 0.911
λ_5	: 0.806		

Table 5.1: SLEIGN2 eigenvalues with $L = 1, j = 5, p = 10, \omega' = 1$.

These eigenvalues are bounded from above by $\omega'^2 = 1$, which is the asymptotic limit of the potential (5.78). This means that the spectrum accumulates at $\lambda = 1$. We have truncated the computation at λ_{10} whose value is relatively close to the accumulation point.

We can adjust the spectrum to the energy levels of the Coulomb problem [55],

$$-\psi'' + \left(-\frac{\alpha}{r} + \frac{l(l+1)}{r^2}\right)\psi = E\psi, \tag{5.92}$$

given by

$$E_n = -\frac{\alpha^2}{4(n+l+1)}, \quad n = 0, 1, 2 \dots. \tag{5.93}$$

So a comparison with (5.83) shows that $\alpha = 8$ and $l = 3.14$. Then setting $\lambda_n = 1 + E_n$ we get the approximation

$$\lambda_n = 1 - \frac{16}{(n+4.14)^2}, \tag{5.94}$$

which is a good fit for the spectrum of table 5.1 as we can see in figure 5.2.

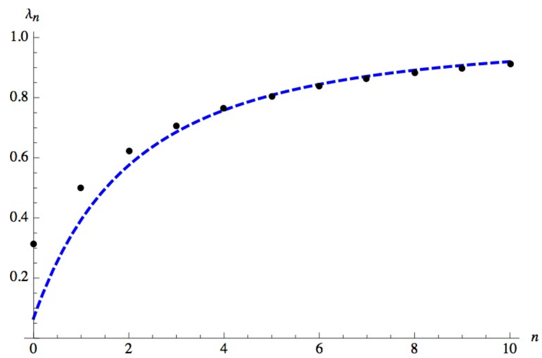

Figure 5.2: Plot of spectrum of table 5.1 and formula (5.94) (dashed blue line).

Chapter 6

Conclusion

The work in this thesis began as an investigation of the zero-modes of the Dirac operator as a possible model for the spin degrees of freedom of particles, motivated by GMM where fundamental particles are modelled by gravitational instantons. In the TN model for the electron, the kernel of the Dirac operator, twisted by a $U(1)$ connection with charge p has a dimension which grows quadratically with $[p]$, as first pointed out by Pope. Our approach to compute the kernel is a novel one and is based on a direct computation of the zero-modes. Using this, we have been able to show that the kernel decomposes into irreducible $SU(2)$ representations up to dimension $[p]$. Another interesting result is that by picking $p > 0$, we obtain a normalisable spin-$\frac{1}{2}$ doublet provided $p > 2$, but one automatically gets a spin-0 state too, which we cannot eliminate by a natural condition.

Our discussion shows that the inclusion of the magnetic field, given by an abelian gauge potential, preserves the dynamical symmetry of the phase space of the TN model. This enabled us to use the conserved angular momentum and Runge-Lenz vector to study the classical and quantum dynamics in the gauged case. Previous work in this area came out in [3], where it is shown that classical elliptical orbits and quantum bound states can occur in the non-gauged case if $L < 0$. We found that even though there are neither bounded orbits nor bound states in the case $L > 0$, the magnetic field produces both.

The toy model of Sect. 4.2, describing the magnetic binding of trajectories on a

2-dimensional cigar-shaped surface, provides a nice qualitative explanation for the existence of bounded orbits and quantum bound states on the TN manifold. In this picture, the restriction of the magnetic field to the cigar-shaped submanifolds of the TN space acts as 'magnetic plug' which keeps trajectories in a bounded region and produces quantum bound states, provided the energy is sufficiently small and $|q| < |p/2|$. A link of this problem with Landau states, explained in [5], is that in the limit when the TN parameter ϵ in (3.100) is taken to zero, the TN problem actually becomes a 4-dimensional Landau problem.

In the computation of the bound states of the Laplace operator, we have separated variables by expanding the solution in terms of a new set of complex functions of the Wigner-type. For the cross sections of the scattering solutions, we have found an interesting electric-magnetic duality between the special cases $s = 0$ and $p = 0$. The gauged version of the Runge-Lenz vector enabled us to re-derive the energy spectrum algebraically and to interpret its degeneracy. By using a twistor formulation of phase space, we showed that the angular momentum and the gauged Runge-Lenz vectors are conserved quantities of a $SU(2,2)$ symmetry.

Using the results from Chapter 3, we have been able to compute the zero-modes on the ES geometry twisted by an abelian gauge potential of charge p. We found that the zero-modes decompose into $|p|$ copies of irreducible $SU(2)$ representations of dimension $|p|$. The study of the bound states of the Laplace operator on ES turned out to be harder than in the TN case. The reason for this is that, conversely to the TN case, the Laplace operator cannot be diagonalised, giving a system of coupled differential equations for the spinor components. As a starting point, we therefore considered scalar fields only and computed the eigenvalues of the twisted Laplace-Beltrami operator by using the open code SLEIGN2. In this case, the existence of scalar bound states can also be explained by the binding of a magnetic field restricted to a 2-dimensional submanifold of ES provided $|L\omega'| < |p/2|$.

We have taken the first steps towards the description of spin in the geometric models for the electron and neutron given by the TN and ES manifolds, and shown that the inclusion of a $U(1)$ gauge potential provides the necessary states for spin-$\frac{1}{2}$. However, further work is needed to explain the extra states that automatically emerge. Perhaps the exploration of spin in other particle candidates will provide more insight into this issue. An interesting case is the Atiyah-Hitchin model for the proton where the Dirac operator, coupled to a $U(1)$ gauge potential of charge n, is known [29] to have a kernel of dimension 2 when $n = 4$. Another interesting case that remains to be studied is the Taub-Bolt model for the proton.

Appendix A

Commutation relations

This appendix lists the commutation relations, involving the gauged angular momentum and Runge-Lenz vectors, which are used in the main text in the algebraic derivation of the energy levels of TN. We do not include the analogous relations of the non-gauged case since they are already known - see for example [41].

A.1 Computation of $[M_k^p, M_l^p]$

In order to compute the commutator $[M_k^p, M_l^p]$ of the gauged Runge-Lenz vector (4.210) we use the analogous relation of the non-gauged case [41],

$$[M_k, M_l] = i\left(\frac{q^2}{L^2} - H\right)\epsilon_{klm}J_m. \tag{A.1}$$

Using this and recalling that

$$\vec{M}^p = \vec{M} - \vec{f} \tag{A.2}$$

with $\vec{f}$ given in (4.122), we then see that

$$\begin{aligned}
[M_k^p, M_l^p] &= [M_k - f_k, M_l - f_l] \\
&= [M_k, M_l] - [M_k, f_l] + [M_l, f_k], \\
&= i\left(\frac{q^2}{L^2} - H\right)\epsilon_{klm}J_m - \frac{p^2}{8L}\left([M_k, \frac{x_l}{rV}] - [M_l, \frac{x_k}{rV}]\right).
\end{aligned} \tag{A.3}$$

In order to work out the second term we use the following computation

$$[M_k, \frac{x_l}{rV}] = \epsilon_{kab} p_a [J_b, \frac{x_l}{rV}] + \epsilon_{kab}[p_a, \frac{x_l}{rV}] J_b - i[p_k, \frac{x_l}{rV}] - \frac{L x_k}{2r}[H, \frac{x_l}{rV}]. \tag{A.4}$$

Note that the third term in the preceding relation is symmetric, and hence it does not contribute to the commutator $[M_k^p, M_l^p]$. A direct computation shows that the other three terms are

$$\begin{aligned} \epsilon_{kab} p_a [J_b, \frac{x_l}{rV}] &= i p_n \frac{x_n}{rV}\delta_{kl} - i p_l \frac{x_k}{rV} \\ &= i p_n \frac{x_n}{rV}\delta_{kl} - i \frac{x_k}{rV} p_l - i[p_l, \frac{x_k}{rV}], \end{aligned} \tag{A.5}$$

$$\epsilon_{kab}[p_a, \frac{x_l}{rV}] J_b = -\frac{i}{rV}\epsilon_{klb} J_b - i\frac{x_l}{rV^2} p_k + i\frac{x_k x_l}{r^3 V^2} x_j p_j, \tag{A.6}$$

$$-\frac{L x_k}{2r}[H, \frac{x_l}{rV}] = \frac{i L x_k}{r^2 V^2} p_l - \frac{L x_k x_l}{r^4 V^3} - i\frac{L x_k x_l}{r^4 V^3} x_n p_n - \frac{L^2 x_k x_l}{r^5 V^4}. \tag{A.7}$$

Then

$$\begin{aligned} [M_k, \frac{x_l}{rV}] - [M_l, \frac{x_k}{rV}] &= \left(-\frac{i}{rV} + \frac{i}{rV^2} + \frac{iL}{r^2V^2}\right)(x_k p_l - x_l p_k) - \frac{2i}{rV}\epsilon_{klm} J_m \\ &= -\frac{2i}{rV}\epsilon_{klm} J_m. \end{aligned} \tag{A.8}$$

So altogether

$$\begin{aligned} [M_k^p, M_l^p] &= i\left(\frac{q^2}{L^2} - H\right)\epsilon_{klm} J_m + \frac{i p^2}{4LrV}\epsilon_{klm} J_m \\ &= i\left(\frac{q^2}{L^2} + \frac{p^2}{4LrV} - H\right)\epsilon_{klm} J_m \\ &= i\left[\frac{1}{L^2}\left(q + \frac{p}{2}\right)^2 - H_p\right]\epsilon_{klm} J_m, \end{aligned} \tag{A.9}$$

where we have used the relation (4.113) in the last step.

A.2 Computation of $\vec{M}^p \cdot \vec{J}$

Using again (A.2) we observe that

$$\begin{aligned} M_k^p J_k &= \left(M_k - \frac{p^2 x_k}{8LrV}\right) J_k \\ &= M_k J_k - \frac{qp^2}{8LV}. \end{aligned} \tag{A.10}$$

A direct substitution of (4.209) gives for the first term on the right hand side

$$M_k J_k = \epsilon_{klm} p_l J_J J_k - i p_k J_k - \frac{x_k}{r}\left(\frac{LH}{2} - \frac{q^2}{L}\right) J_k. \tag{A.11}$$

Using the fact that $\vec{J}$ commutes with the Hamiltonian and employing the identity

$$\epsilon_{lmk} J_m J_k = i J_l, \tag{A.12}$$

which follows from (4.208), we get

$$M_k J_k = qL\left(\frac{q^2}{L^2} - \frac{1}{2}H\right). \tag{A.13}$$

Thus

$$\begin{aligned} M_k^p J_k &= L\left(\frac{q^2}{L^2} - \frac{1}{2}H\right) - \frac{qp^2}{8LV} \\ &= -q\left(\frac{1}{2}LH - \frac{q^2}{L} + \frac{p^2}{8LV}\right) \\ &= -q\left(\frac{1}{2}LH_p - \frac{q^2}{L} - \frac{pq}{2L}\right). \end{aligned} \tag{A.14}$$

Similarly it follows that $M_k^p J_k = J_k M_k^p$.

A.3 Computation of $\vec{M}^p \cdot \vec{M}^p$

Using (A.2) again, we see that

$$\begin{aligned}
M_k^p M_k^p &= \left(M_k - \frac{p^2 x_k}{8LrV}\right)\left(M_k - \frac{p^2 x_k}{8LrV}\right) \\
&= M_k M_k - \frac{p^2}{8L} M_k \frac{x_k}{rV} - \frac{p^2}{8L}\frac{x_k}{rV} M_k + \frac{p^4}{64L^2V^2} \\
&= M_k M_k - \frac{p^2}{4LrV} x_k M_k - \frac{p^2}{8L}[M_k, \frac{x_k}{rV}] + \frac{p^4}{64L^2V^2}, \qquad \text{(A.15)}
\end{aligned}$$

where we have used $M_k \frac{x_k}{rV} = \frac{x_k}{rV} M_k + [M_k, \frac{x_k}{rV}]$ in the last step. The second term on the right hand side is

$$\begin{aligned}
-\frac{p^2}{4LrV} x_k M_k &= -\frac{p^2}{4LrV}\left[\epsilon_{klm} x_k p_l J_m - i x_k p_k - r\left(\frac{1}{2}LH - \frac{q^2}{L}\right)\right] \\
&= -\frac{p^2}{4LrV}(J_m J_m - q^2) + \frac{ip^2}{4LrV} x_k p_k + \frac{p^2}{4LV}\left(\frac{1}{2}LH - \frac{q^2}{L}\right). \qquad \text{(A.16)}
\end{aligned}$$

To compute the third term we use the result

$$\begin{aligned}
[M_k, \frac{x_l}{rV}] &= i p_n \frac{x_n}{rV}\delta_{kl} - i\frac{x_k}{rV} p_l - i[p_l, \frac{x_k}{rV}] - \frac{i}{rV}\epsilon_{klb} J_b - i\frac{x_l}{rV^2} p_k + i\frac{x_k x_l}{r^3V^2} x_j p_j \\
&\quad + \frac{iLx_k}{r^2V^2} p_l - \frac{Lx_k x_l}{r^4V^3} - i\frac{Lx_k x_l}{r^4V^3} x_n p_n - \frac{L^2 x_k x_l}{r^5V^4} - i[p_k, \frac{x_l}{rV}], \qquad \text{(A.17)}
\end{aligned}$$

which in the case $l = k$ reduces to

$$[M_k, \frac{x_k}{rV}] = \frac{2i}{rV} x_l p_l + \frac{2}{rV} + \frac{L}{r^2V^2} + \frac{iL}{r^2V^2} x_l p_l - \frac{L}{r^2V^3} - \frac{iL}{r^2V^3} x_l p_l - \frac{L^2}{r^3V^4}. \qquad \text{(A.18)}$$

Multiplying this by $-p^2/8L$ and simplifying, we obtain for the third term

$$\begin{aligned}
-\frac{p^2}{8L}[M_k, \frac{x_k}{rV}] &= -\frac{ip^2}{4LrV} x_l p_l - \frac{p^2}{4LrV} - \frac{p^2}{8r^2V^2} - \frac{ip^2}{8r^2V^2} x_l p_l \\
&\quad + \frac{p^2}{8r^2V^3} + \frac{ip^2}{8r^2V^3} x_l p_l + \frac{Lp^2}{8r^3V^4}, \\
&= -\frac{ip^2}{4LrV} x_l p_l - \frac{p^2}{4LrV} - \frac{iLp^2}{8r^3V^3} x_l p_l - \frac{L^2p^2}{8r^4V^4}. \qquad \text{(A.19)}
\end{aligned}$$

We then observe that the sum of the second and third terms gives

$$-\frac{p^2}{4LrV} x_k M_k - \frac{p^2}{8L}[M_k, \frac{x_k}{rV}] = -\frac{p^2}{4LrV}(J_m J_m - q^2 + 1) + \frac{p^2}{8LV} K + K\frac{p^2}{8LV}, \qquad \text{(A.20)}$$

where $K := \frac{1}{2}LH - \frac{q^2}{L}$. Finally, using the above along with the result [41],

$$M_k M_k = \left(H - \frac{q^2}{L^2}\right)(J_m J_m - q^2 + 1) + K^2, \tag{A.21}$$

in (A.15), we obtain

$$\begin{aligned} M_k^p M_k^p &= (H - \frac{q^2}{L^2})(J_m J_m - q^2 + 1) - \frac{p^2}{4LrV}(J_m J_m - q^2 + 1) \\ &+ K^2 + \frac{p^2}{8LV}K + K\frac{p^2}{8LV} + \frac{p^4}{64L^2V^2}, \\ &= \left(H - \frac{q^2}{L^2} - \frac{p^2}{4LrV}\right)(J_m J_m - q^2 + 1) + \left(K + \frac{p^2}{8LV}\right)^2 \\ &= \left[H_p - \frac{1}{L^2}\left(q + \frac{p}{2}\right)^2\right](J_m J_m - q^2 + 1) + \left(\frac{1}{2}LH_p - \frac{q^2}{L} - \frac{pq}{2L}\right)^2. \end{aligned} \tag{A.22}$$

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